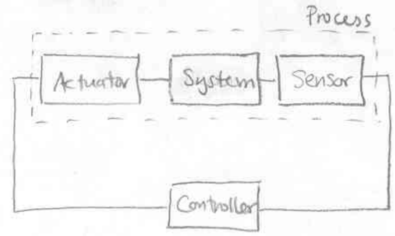
10/2 W

Midterm Review (Lecture 5-3)

In the beginning we talked, conceptually, about control systems in the following form:



So far we've almost exclusively been interested in just the process (w/o feedback, w/o control). We've asked question like:

- How many equilibrium points does it have?
- Are these eq. points stable?

These are asked (so far) about the system without control/feedback

Recently we've started to talk about a special kind of control and feedback (state feedback), but most of what we've done so far is analysis of the dynamics of the process.

Converting to State Space Form

- if already a set of 1st order ODEs then convert to vector form.
- if a higher order ODE w/ constant coefficients then convert to a series of 1st order ODEs:

$$z^{(n)} + a_{n-1} z^{(n-1)} + \dots a_1 z + a_0 = 0$$

$$x_1 = z \qquad x_2 = \dot{z} \qquad x_3 = \ddot{z} \quad \dots \quad x_n = z^{(n-1)}$$

Our goal is to get the system in the form:

$$\dot{\underline{x}} = f(\underline{x}, \underline{u}) \leftarrow \text{dynamics depend on state and input}$$

$$y = h(x, u)$$

$\underline{x} \in \mathbb{R}^n$ state
$\underline{u} \in \mathbb{R}^m$ input
$y \in \mathbb{R}^p$ output

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Finding eq. points

Once the system is in state space form we can find the equilibrium points by setting $\dot{\underline{x}} = 0$

$$0 = f(\underline{x}_{eq}, \underline{u}_{eq})$$

Note: if $f(\underline{x}, \underline{u})$ is linear there is only one equilibrium point. Nonlinear systems can have multiple equilibrium points.

e.g. $\ddot{z} = z - z^2$

convert to state space

$$x_1 = z \qquad \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 - x_1^2 \end{pmatrix}$$
$$x_2 = \dot{z} \qquad \dot{\underline{x}} = f(\underline{x}, \underline{u})$$

find eq. points

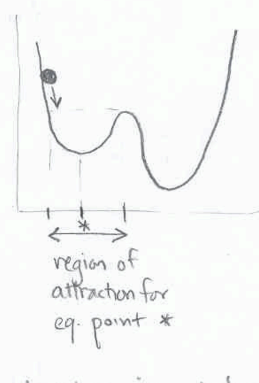
$$0 = x_2$$
$$0 = x_1 - x_1^2 = x_1(1 - x_1)$$

$(x_1, x_2) \rightarrow (0,0)$ and $(1,0)$ 2 eq. points.

Regions of Attraction / Local vs. Global Stability

Equilibrium points for NL systems can have regions of attraction

conceptual example:



linear systems: local behavior is global behavior

nonlinear systems: analysis of eq. points will only tell you local behavior.

Midterm Review

10/26/06
MJD

Phase Portraits

Can be drawn for 2D systems. $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Plots x_1 vs. x_2.

Useful for determining stability of eq point, but tells nothing about time.

Linearizing

Basically all physical systems are nonlinear, but linearization can be very useful.

ex. balancing

feedback control keeps system in linear regime.

$$\dot{\underline{x}} = f(x, \underline{u}) \qquad A = \left.\frac{\partial f}{\partial x}\right|_{x_{eq}, u_{eq}} \qquad B = -\left.\frac{\partial f}{\partial u}\right|_{x_{eq}, u_{eq}}$$

e.g. $\ddot{z} = z - z^2$

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 - x_1^2 \end{pmatrix} \longrightarrow \left.\begin{pmatrix} 0 & 1 \\ 1-2x_1 & 0 \end{pmatrix}\right|_{x_{eq}}$$

about $(0,0)$ $\quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

about $(1,0)$ $\quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 - 1 \\ x_2 \end{pmatrix}$

Linear Systems

In general we can write a linear system as

$$\dot{\underline{x}} = Ax + Bu$$
$$y = Cx + Du$$

The solution to this is:

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)$$

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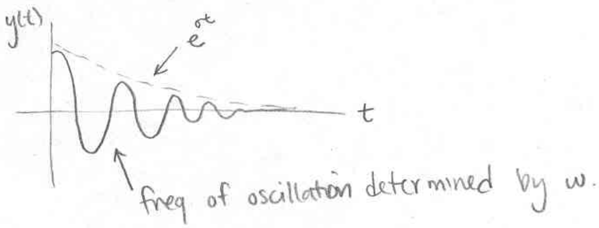
Recall that e^{At} determines behavior of $y(t)$ as $t \to \infty$.

We proved, using Jordan form, and the matrix exponential that the eigenvalues of A determine stability of the system.

e.g. 2D system

complex e.vals

$\lambda_{1,2} = \sigma \pm i\omega$



note: $\sigma < 0 \rightarrow y(t) \rightarrow 0$ (asy. stable)

$\sigma = 0 \rightarrow y(t)$ remains bounded (stable)

$\sigma > 0 \rightarrow y(t) \rightarrow \infty$ (unstable)

Phase Portrait Classification of Eq. Points

(see photocopied notes)

Lyapunov Functions

Useful (if you can find one) for proving stability of nonlinear systems.

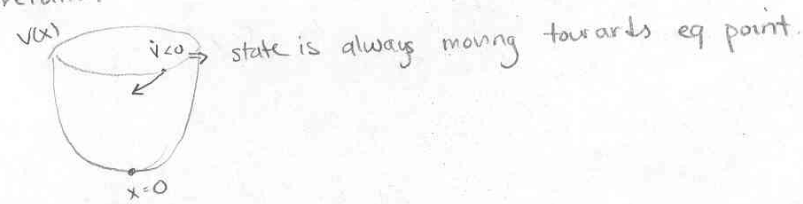
$V(x) = 0$ at $x = 0$

$V(x) > 0$ at $x \neq 0$

In some neighborhood of $x = 0$ (the eq point)

$\dot{V}(x) < 0$ for $x \neq 0 \Rightarrow x = 0$ is an asymptotically stable eq. point

Physical interpretation

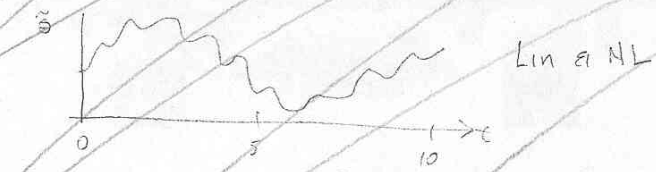


CDS 110 a Section Notes 10/20/05

Extra Office Hours, HW out early

Comparing Lin vs. NL pendulum from Lecture Notes

$\omega = 10$



Does this look weird to you? Given that last week I told you

$\sin \omega t \rightarrow \boxed{\text{Lin}} \rightarrow A \sin \omega t$

Answer: Start up transients

Types of Equilibrium Points : $x \in \mathbb{R}^2$

E-vals	$\dot{\underline{x}} = A\underline{x}$		$\dot{\underline{x}} = f(\underline{x})$	
λ complex $Re[\lambda] < 0$	spiral	(globally) asymptotically stable	spiral	(locally) asymptotically stable
λ real or λ complex $Re[\lambda] = 0$	center	stable (aka stable ISL)	?	H.O.T's matter need to use another method. e.g.: Lyapunov Function - Hamiltonian systems $\ddot{x} = -4x^3 + 2x$
λ complex $Re[\lambda] > 0$	spiral	unstable		unstable
λ real $\lambda < 0$	sink	(globally) stable	sink	(locally) stable
one $Re[\lambda] < 0$ one $Re[\lambda] > 0$	saddle	unstable	saddle	unstable
λ real $\lambda > 0$	source	unstable	source	unstable

Midterm Review

State Feedback

Special kind of control law

$$u = -Kx$$

$$\dot{x} = Ax + Bu = (A - BK)x$$

eigenvalues of $A-BK$ now govern stability of system.

Can place these e-vals <u>anywhere</u> if the system is reachable.

(neg. real parts would be a good choice, we'll talk more about design considerations in the second half of the term).

Reachability

Can check if system is reachable by constructing

$$W_r = [B \;\; AB \;\; A^2B \; \ldots \; A^{n-1}B]$$

if W_r is full rank $\Rightarrow$ system is reachable

if W_r is square can test rank by taking determinant

$\det(W_r) = 0 \Rightarrow$ not full rank $\Rightarrow$ not reachable

$\det(W_r) \neq 0 \Rightarrow$ full rank $\Rightarrow$ reachable