Final Review 12/7/07

Outline

1) Topical review

2) Extended example

I) Course Review

1) Models, stability, linear systems

We started the course talking about how to model physical systems as systems of differential equations, these systems not necessarily being linear.

We then talked about the behavior of these systems and analysis that could be completed on the models, specifically stability.

a) equilibrium point → $\dot{x} = f(x)$

x_e are points $\ni f(x_e) = 0$

stationary points for dynamics

i) stable, asymptotically stable, or unstable (draw pics)

b) ways to determine stability

i) phase plot

ii) Lyapunov fcn $V(x) > 0$, $\dot{V}(x) < 0, \forall x$

where $\dot{V}(x) = \frac{dV}{dx}\frac{dx}{dt} = \frac{dV}{dx} f(x)$ $(\dot{x} = f(x))$

iii) eigenvalues of A matrix (if not 0) $\lambda_i < 0$ $\forall i \Rightarrow$ stable

gives local stability for linearized systems.

c) How to linearize around an equilibrium point?

$\dot{x} = f(x, u)$ where $0 = f(x_e, u_e)$

Take Taylor series around x_e, u_e:

$$\dot{x} = \underbrace{f(x_e, u_e)}_{\phi} + \underbrace{\frac{\partial f}{\partial x}\Big|_{(x_e,u_e)}}_{A} \underbrace{(x - x_e)}_{z} + \underbrace{\frac{\partial f}{\partial u}\Big|_{(x_e,u_e)}}_{B} \underbrace{(u-u_e)}_{v} + \underbrace{HOT}_{\text{ignore}}$$

$\Rightarrow \underline{\dot{z} = Az + Bv}$ (if $u_e = 0 \Rightarrow v = u$, etc)

linear approximation, good near the equilibrium point (when $x - x_e$ is small)

$\Rightarrow$ like small angle approximation

2) Reachability & State Feedback

a) Reachability - ability to place poles arbitrarily

Test: $[B \quad AB \quad A^2B \cdots]$ is full rank ($\det \neq 0$)

b) Pole placement: ability of state feedback to alter characteristic equation of system:

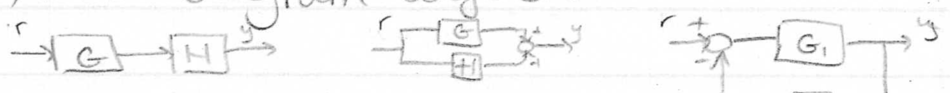
$T = \frac{PC}{1+PC}$

Design C such that can place poles ($0 = 1 + PC$) where desired.

3) Transfer Functions

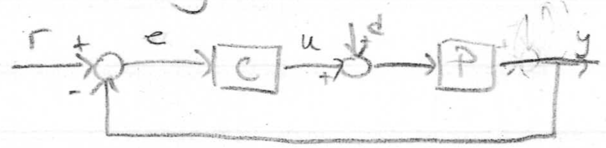
- Tool for analyzing systems using freq. resp.

a) Block diagram algebra:



$H_{yr} = GH$ $\qquad$ $H_{yr} = G + H$ $\qquad$ $H_{yr} = \frac{G_1}{1 + G_1 G_2}$

b) Finding tf from block diagrams



ex: Find H_{yr}

$y = P(d+u)+n$

$y = Pd + PCe + n$

$y = Pd + PC(r-y) + n$

$(1+PC)y = Pd + PCr$

$y = \frac{P}{1+PC} d + \frac{PC}{1+PC} r$

$= H_{yd} \cdot d + H_{yr} \cdot r$

$\Rightarrow H_{yr} = \frac{PC}{1+PC}$

! Remember gang of four?

S = sensitivity = $\frac{1}{1+PC}$ = H_{er}

T = complementary sensitivity = $\frac{PC}{1+PC}$ = H_{yr}

PS = load sensitivity = $\frac{P}{1+PC}$ = H_{yd}
(disturbance)

CS = noise sensitivity = $\frac{C}{1+PC}$

4) Frequency Response & Analysis

- Big concentration (w/ design) of the 2nd half of this class.

a) Bode plots

- draw from tf
- rules:

1) each LHP pole causes slope of -1 in mag, -90 in phase

2) pair of complex poles w/ neg real part cause slope of -2 at $\omega_d = \omega_0\sqrt{1-\zeta^2}$, phase of -180

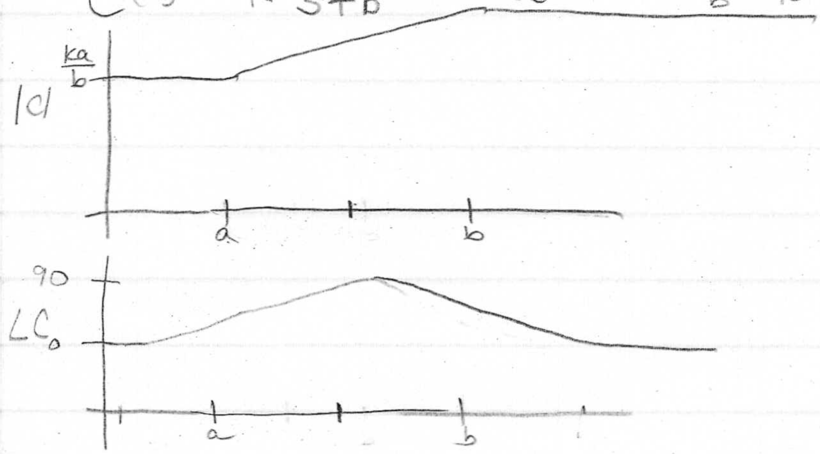
3) zeros (LHP) give +1 slope in mag (at zero) & +90 in phase

4) RHP poles/zeros same mag, start −180 & opposite phase

5) zero freq gain: Plug $s=0$ into tf, value is gain

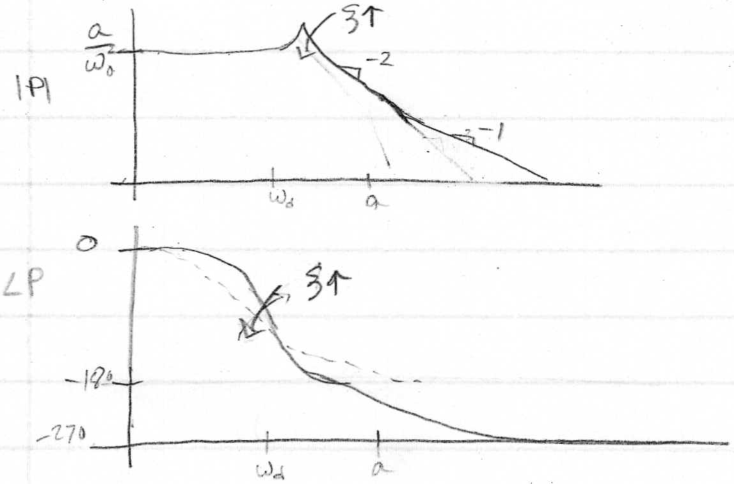
Draw some:

– $C(s) = K\frac{s+a}{s+b}$ $a<b$ $b=100a$



Lead controller: notice increase in gain after (potentially) initial decrease (if $K < 100$). Also, bump in phase. Tf mult ≡ Bode addition

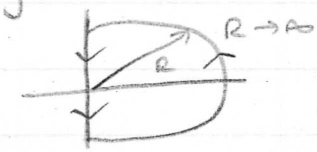
– $P(s) = \frac{-(s-a)}{s^2+2\zeta\omega_0 s+\omega_0^2}$ $\omega_0 < a$



★ Notice start at 0 phase ... due to leading negative sign.

b) Nyquist

- Can draw these 2 ways:
  - i) From loop tf & contour



  - ii) From Bode plot
    - Remember to switch arrows!
- Test for stability of closed loop system (though use open loop to draw!)
- Criterion:

  P = # RHP poles of $L(s)$

  N = # ccw encirclements of -1

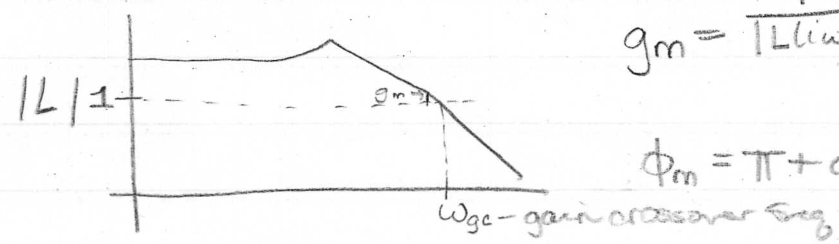
  Z = # RHP zeros of $1+L(s)$ (aka RHP poles of closed loop)

  Then $Z = N + P$

  For stable system, want $Z=0$, $\Rightarrow N=-P$

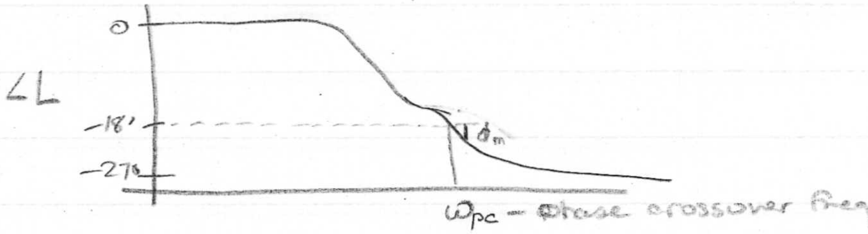
c) Frequency Response Design.

i) Bode



$$g_m = \frac{1}{|L(i\omega_{pc})|}$$

$$\phi_m = \pi + \arg(L(i\omega_{gc}))$$

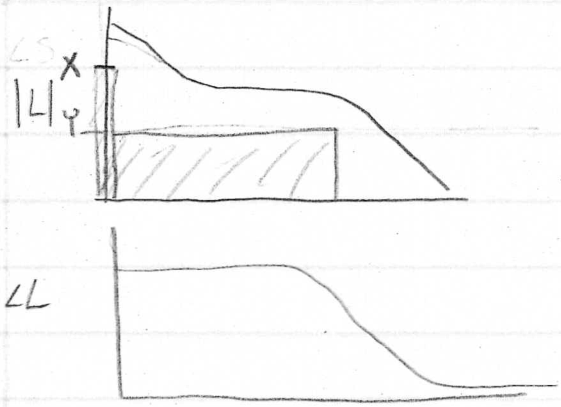


SS error: $\frac{1}{1+L(0)} < x\%$

$\Rightarrow L(0) > \frac{1-x}{x} = X$

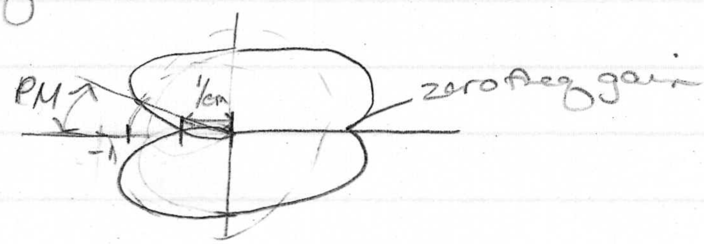
tracking error: $\frac{1}{1+L} < \cancel{x}\, y\%$

$\Rightarrow L > \frac{1-\cancel{x}\,y}{\cancel{x}\,y} = Y$



Bandwidth: closed loop Bode, decrease of $1/\sqrt{2} \times$ zero freq gain (-3dB)

ii) Nyquist



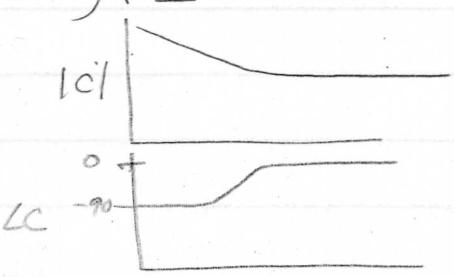
5) Controller Design & Considerations

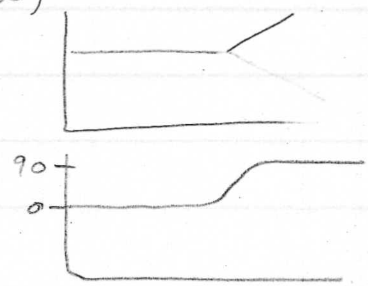
1) PID controllers

i) proportional - just displaces gain by k_p

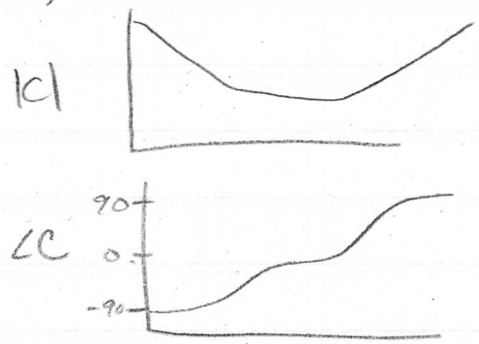
ii) PI

iii) PD





iv) PID



* Already did lead.

- can do first cut by Ziegler-Nichols

v) on step response.

b) Design limitations

- want to keep $S = \frac{1}{1+L}$ & $T = \frac{L}{1+L}$ small,
  - S: low tracking error
  - T: good robustness

however $S + T = 1$

$\omega \downarrow$, S small, $\omega \uparrow$, T small

- Bode Integral formula:

area under sensitivity fcn is conserved (in dB)

$\int_0^\infty \ln \frac{1}{|1+L(j\omega)|} d\omega = \pi \sum \mathrm{Re}\, p_k$

p_k: $1 + L(p_k) = 0$

c) Performance, Robust Stability

- $|W_1 S| < 1$

$W_1 S$: measure of performance (ss error, tracking error

$|S| < .1 \Rightarrow W_1 = 10$

- $|W_2 T| < 1$

$W_2 T$: measure of stability (like pm, gm, though not as clear how to convert)

- Robust performance

$|W_1 S| + |W_2 T| < 1$

Just a combo of the two!

Some constraints on gains, fcns W_1, W_2

because of $S+T=1$ constraint.

Ensures all constraints are satisfied for all frequencies.

II) Example

Have system:

$$\dot{x}_1 = .5x_1 + x_1^2 + x_2$$

$$\dot{x}_2 = 2x_2 + u$$

$$y = x_1$$

a) Linearize, about $(x_1, x_2, u) = (0,0,0)$ state space, give transfer function H_{yu},

b) Bode of plant

c) Design controller for ss error < 5%, pm more than 45°. Show Nyquist of open loop system.

Solution:

a) $\dot{x}_1 = f_1(x_1, x_2, u) = .5x_1 + x_1^2 + x_2$

$\dot{x}_2 = f_2(x_1, x_2, u) = 2x_2 + u$

$$A = \begin{bmatrix} \left.\frac{\partial f_1}{\partial x_1}\right|_{(0,0,0)} & \left.\frac{\partial f_1}{\partial x_2}\right|_{(0,0,0)} \\ \left.\frac{\partial f_2}{\partial x_1}\right|_{(0,0,0)} & \left.\frac{\partial f_2}{\partial x_2}\right|_{(0,0,0)} \end{bmatrix} = \begin{bmatrix} .5 & 1 \\ 0 & 2 \end{bmatrix}$$

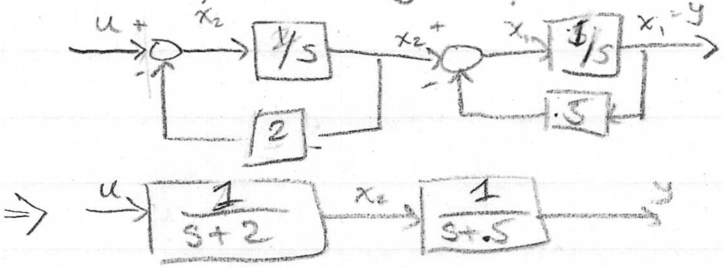
$$B = \begin{bmatrix} \left.\frac{\partial f_1}{\partial u}\right|_{(0,0,0)} \\ \left.\frac{\partial f_2}{\partial u}\right|_{(0,0,0)} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$C = [1 \quad 0]$ $\quad D = 0$ $\quad$ (by inspection)

Find tf: 2 ways

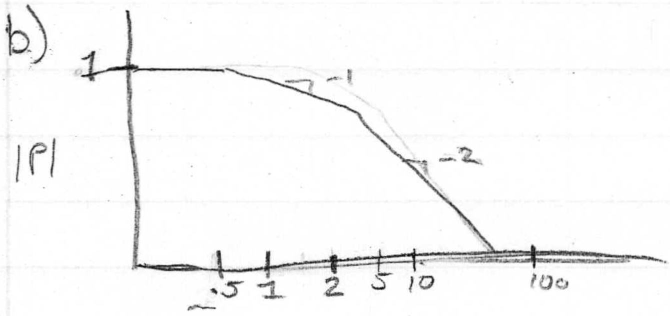
1) $C(sI-A)^{-1}B$

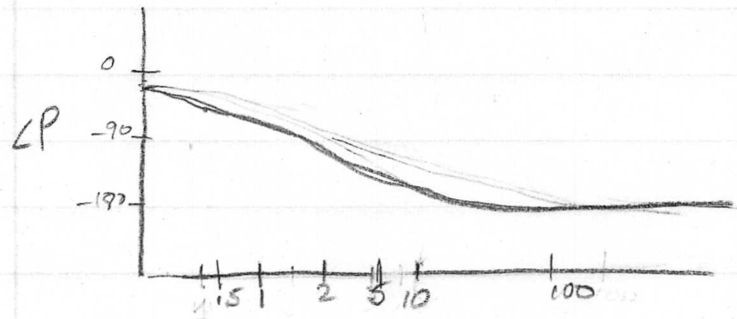
2) Block diagram



$$H_{yu} = \frac{1}{(s+2)(s+.5)} = P(s)$$

b)





c) SS error: $\frac{1}{1+L(0)} < .05$

$$\Rightarrow L(0) = \frac{.95}{.05} = 19 \approx 26\text{ dB}$$

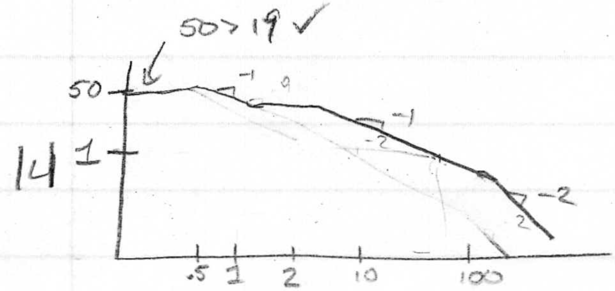
Try proportional controller of about 50, see if need phase lead.

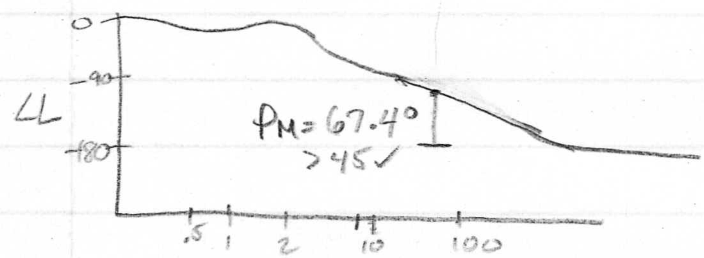
From Matlab, get margin of ~20.2 @ 7 rad/s

$\Rightarrow$ need lead controller!

Let $C(s) = 50 \cdot 100 \frac{s+1}{s+100} = 5000 \frac{s+1}{s+100}$

$L(s) = \frac{5000(s+1)}{(s+100)(s+.5)(s+2)}$





Draw Nyquist to ensure stability