Lecture 4.2: Linear Systems Analysis

Today

1. Matrix exponential
2. Jordan form
3. Stability of linear ODEs
4. Impulse response and convolution
5. Linearization

Matrix exponential

Claim $x(t) = e^{At} x_0$ is the solution of $\dot{x} = Ax$, $x(0) = x_0$

Pf By defn, $e^{At} = I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \dots$

Differentiating $x(t)$, we have

$$\begin{aligned}
\dot{x} &= \frac{d}{dt}\left(e^{At} x_0\right) \\
&= \frac{d}{dt}\left(I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \dots\right) x_0 \\
&= \left(0 + A + A^2 t + \frac{1}{2!} A^3 t^2 + \dots\right) x_0 \\
&= A\left(I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \dots\right) x_0 \\
&= A\left(e^{At}\right) x_0 = Ax \;/\!/
\end{aligned}$$

To explore the behavior of $\dot{x} = Ax$, we must understand the structure of e^{At}. Specifically:

- when does $e^{At} \to 0$ as $t \to \infty$ (stability)
- what directions does e^{At} grow/shrink fastest (stability)

Jordan form

Suppose we change coordinates and write $z = Tx$ where $T \in \mathbb{R}^{n \times n}$ is an invertible matrix

$$\dot{z} = T\dot{x} = TAx = TAT^{-1}z$$

If $x=0$ is stable in x coordinates then $z=0$ stable in z coords $\Rightarrow$ these systems are equivalent. The transformation $A \mapsto TAT^{-1}$ is called a similarity transformation.

Q: Can we find $T \in \mathbb{C}^{n \times n}$ that makes TAT^{-1} particulary simple/useful

A: Jordan form: $J = TAT^{-1}$

$$J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ & J_2 & & \vdots \\ & & \ddots & 0 \\ 0 & & & J_q \end{bmatrix} \qquad J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & \lambda_i \end{bmatrix} \in \mathbb{R}^{m_i \times m_i}$$

λ_i = ith eigenvalue of A

m_i = size of the ith Jordan block

Remarks

1. Jordan form is unique (up to permutation in blocks)
2. Can have multiple blocks w/ same eigenvalue
3. Use this to compute matrix exponential

$$e^{At} = e^{(T^{-1}JT)t} = T^{-1}e^{Jt}T = T^{-1}\begin{bmatrix} e^{J_1} & & 0 \\ & \ddots & \\ 0 & & e^{J_q} \end{bmatrix} T$$

The exponential of a Jordan block determines behavior of a portion of a linear system:

$$e^{J_i t} = e^{\lambda_i t}\begin{bmatrix} 1 & t & \frac{1}{2}t^2 & \cdots & \frac{1}{m!}t^{m_i-1} \\ & 1 & t & \cdots & \frac{1}{(m-1)!}t^{m_i-2} \\ & & \ddots & & \vdots \\ 0 & & & & 1 \end{bmatrix}$$

If $\mathrm{Re}(\lambda_i) < 0 \Rightarrow e^{J_i t} z_0 \to 0$

If $\mathrm{Re}(\lambda_i) > 0 \Rightarrow e^{J_i t} z_0 \to \infty$

If $\mathrm{Re}(\lambda_i) = 0 \Rightarrow e^{J_i t} z_0 = \begin{bmatrix} 1 & t & \cdots & \frac{1}{m!}t^{m_i-1} \\ & & & \vdots \\ & & & 1 \end{bmatrix} z_0 \to \infty$

So non-trivial Jordan blocks give unstable behavior when $\mathrm{Re}(\lambda_i) = 0$.

Determining the size of Jordan blocks

The size of the Jordan blocks, m_i, are determined by the characteristic polynomial

$$\lambda(s) = \det(sI - A) = (s-\lambda_1)^{m_1}(s-\lambda_2)^{m_2}\cdots(s-\lambda_q)^{m_q}$$

m_i = algebraic multiplicity

$\dim \mathcal{N}(\lambda I - A)$ = # Jordan blocks w/ eigenvalue λ

$$\dim \mathcal{N}(\lambda I - A)^k = \sum_{\lambda_i = \lambda} \min\{k, n_i\}$$

Using these relationships, we can figure out the length of the Jordan blocks $m_1, \ldots, m_q$

Example

$$J = \left[\begin{array}{c|c|cc} \lambda_1 & & & \\ \hline & \lambda_1 & & \\ \hline & & \lambda_1 & 1 \\ & & & \lambda_1 \end{array}\right]$$

$$(\lambda_1 I - J) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathcal{N}(\lambda_1 I - J) = 3$$

$$(\lambda_1 I - J)^2 = \begin{bmatrix} & 0 & \end{bmatrix} \Rightarrow \mathcal{N}(\lambda_1 I - J)^2 = 4 = \sum \min\{2, m_i\}$$

$$\Rightarrow \quad m_1 + m_2 + m_3 = 4$$

$$m_1 = 1, \; m_2 = 1, \; m_3 = 2$$

(or some permutation)

See ACM 95/100 for more general treatment

Computing T

To compute T, we write

$$T = [T_1 | T_2 | \cdots | T_q] \qquad T_i = [v_{i1} \; v_{i2} \cdots v_{in_i}] \in \mathbb{C}^{n \times n_i}$$

$$T^{-1} A T = J \Rightarrow A T_i = T_i J_i$$

From the form of J_i, we have

$$A v_{i1} = \lambda_i v_{i1} \qquad j = 1 \qquad \text{eigenvector}$$

$$A v_{ij} = v_{ij-1} + \lambda_i v_{ij} \qquad j = 2, \ldots, n_i \qquad \text{generalized eigenvector}$$

Stability of linear ODEs

Thm A linear system $\dot{x} = Ax$ is asymptotically stable iff $Re(\lambda_i) < 0$ for all eigenvalues $\lambda_1, \dots, \lambda_n$ of A

Pf Write $z = Tx$ where $T^{-1}AT = J$ is in Jordan form. Solution is composed of terms of the form $t^k e^{\lambda_i t}$. If $Re(\lambda_i) < 0$ then $t^k e^{\lambda_i t} \to 0$ as $t \to \infty$

Remarks

1. If $Re(\lambda_i) = 0$ then stability depends on Jordan structure for the eigenvalue

$$\dot{x}_1 = x_2 \qquad\qquad \dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 \qquad\qquad \dot{x}_2 = 0$$

$$\lambda = \pm j \qquad\qquad \lambda = \{0, 0\}$$

$$J = \begin{bmatrix} j & 0 \\ 0 & -j \end{bmatrix} \qquad\qquad J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$x_1(t) = a\cos\omega t + b\sin\omega t \qquad\qquad x_1(t) = x_{10} + t\,x_{20}$$
$$x_2(t) = -b\sin\omega t + a\cos\omega t \qquad\qquad x_2(t) = x_{20}$$

<u>Stable</u> (but not asy stable)   <u>Unstable</u>

2. Can also show $x = 0$ is <u>unstable</u> if any λ_i has positive real part.

Solutions of linear ODEs

RMM 20 Oct 04 ~~22 Oct 02~~

<u>Claim</u> $x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau)\,d\tau$

<u>Pf</u> Recall that $\frac{d}{dt}\left(\int_0^t f(t,\tau)\,d\tau\right) = f(t,\tau)\Big|_t + \int_0^t \frac{\partial f}{\partial t}(t,\tau)\,d\tau$

Differentiating the particular soln we have

$$\dot{x} = e^{A(t-\tau)}Bu(\tau)\Big|_t + \int_0^t \frac{\partial}{\partial t}\left(e^{A(t-\tau)}Bu(\tau)\right)d\tau$$

$$= Bu(t) + \int_0^t Ae^{A(t-\tau)}Bu(\tau)\,d\tau$$

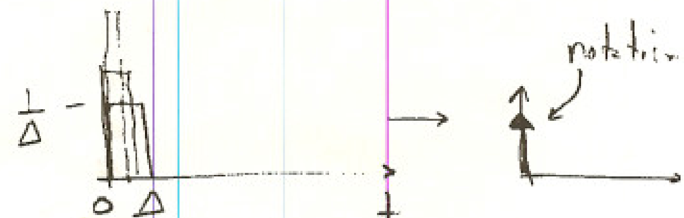
$$= A\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau + Bu = Ax + Bu$$

<u>Remarks</u>

1. If $y = Cx + Du$, get $y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du$
2. $h(t) = Ce^{A(t-\tau)}B$ is called the <u>impulse response</u>:

<u>Defn</u> Let $\delta_\Delta(t)$ be a pulse from $t=0$ to $t=\Delta$ of height $\frac{1}{\Delta}$. The <u>unit impulse</u> $\delta(t)$ is defined as

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$



3. Can interpret ~~view~~ $\int_0^t \cdots$ as the <u>convolution</u> of $h(t)$ & $u(t)$

$$(h * u)(t) = \int_0^t h(t-\tau)\,u(\tau)\,d\tau$$

Intuition: decompose u into "sum" of impulse & <u>superimpose</u> response

RMM 17 Oct 06

(7b)

Linearization

$$\dot{x} = f(x,u) \qquad f(x_e, u_e) = 0$$

$$y = h(x,u) \qquad y_e = h(x_e, u_e)$$

Write Taylor series expansion around x_e, u_e, y_e

$$\dot{x} = \underbrace{f(x_e, u_e)}_{= 0} + \underbrace{\left.\frac{\partial f}{\partial x}\right|_{(x_e,u_e)}}_{A} \underbrace{(x - x_e)}_{z} + \underbrace{\left.\frac{\partial f}{\partial u}\right|_{(x_e,u_e)}}_{B} \underbrace{(u - u_e)}_{v} + \cancel{h.o.t.}$$

ignore

$$\dot{z} = Az + Bv + h.o.t. \leftarrow \text{can ignore for } z \ll 1$$

$$y = h(x_e, u_e) + \underbrace{\left.\frac{\partial h}{\partial x}\right|_{(x_e,u_e)}}_{C} \underbrace{(x - x_e)}_{z} + \underbrace{\left.\frac{\partial h}{\partial u}\right|_{(x_e,u_e)}}_{D} \underbrace{(u - u_e)}_{z}$$

Remarks

1. Same as small angle approximation

Thm The eq pt $x_e = 0$ of (**) is asymptotically stable ($\lim_{k \to \infty} x[k] = 0$) if $|\lambda_i| < 1$ for all eigenvalues $\lambda_i \in \lambda(A)$.

Pf (scalar case)

$$x[k+1] = \lambda x[k]$$

$$x[k] = \lambda^k x[0] \Rightarrow \text{stable if } |\lambda| < 1$$

More general case (diagonalizable): HW #4, problem 3
Most general case: use Jordan decomposition

HW: derive initial condition response and response to input $u[k]$.

Linearization in discrete time

Exactly same as continuous time

$$\begin{aligned} x^+ &= f(x,u) \\ y &= h(x,u) \end{aligned} \quad \longrightarrow \quad \begin{aligned} x^+ &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

$$\left.\begin{aligned} A &= \left.\frac{\partial f}{\partial x}\right|_{x_e, u_e} & B &= \left.\frac{\partial f}{\partial u}\right|_{x_e, u_e} \\ C &= \left.\frac{\partial h}{\partial x}\right|_{x_e, u_e} & D &= \left.\frac{\partial h}{\partial u}\right|_{x_e, u_e} \end{aligned}\right\}$$

To be more careful, redefine

$$z = x - x_e$$
$$v = u - u_e \quad \cdots$$
$$w = y - y_e$$