CDS 202 Winter 2009 Solution Set 8

Problem 1 (MTA 7.1-1)

For $\alpha = 2e^1 \wedge e^3 - e^2 \wedge e^3$ and $\beta = -e^1 + e^2 - 2e^3$, we have

$$\begin{aligned}
\alpha \wedge \alpha &= 2e^1 \wedge e^3 \wedge (2e^1 \wedge e^3) - 2e^1 \wedge e^3 \wedge (e^2 \wedge e^3) - e^2 \wedge e^3 \wedge (2e^1 \wedge e^3) \\
&\quad + e^2 \wedge e^3 \wedge (e^2 \wedge e^3) \\
&= 0, \\
\alpha \wedge \beta &= 2e^1 \wedge e^3 \wedge (-e^1 + e^2 - 2e^3) - e^2 \wedge e^3 \wedge (-e^1 + e^2 - 2e^3) \\
&= 2e^1 \wedge e^3 \wedge e^2 + e^2 \wedge e^3 \wedge e^1 \\
&= -e^1 \wedge e^2 \wedge e^3, \\
\beta \wedge \beta &= 0 \qquad \text{(since } \beta \text{ is a one form)}, \\
\beta \wedge \alpha \wedge \beta &= \beta \wedge \beta \wedge \alpha = 0 \text{ (since } \beta \wedge \beta = 0 \text{ or because any 4-form on } \mathbb{R}^3 \text{ is zero).}
\end{aligned}$$

Problem 2 (MTA 7.1-3)

Let $v_1, ..., v_k$ be linearly dependent vectors. Without loss of generality, assume that v_k can be written as

$$v_k = \sum_{i=1}^{k-1} \beta^i v_i,$$

for some β^i. Then for any $\alpha \in \Lambda^k(E)$, we have

$$\begin{aligned}
\alpha(v_1, ..., v_{k-1}, v_k) &= \alpha\left(v_1, ..., v_{k-1}, \sum_{i=1}^{k-1} \beta^i v_i\right) \\
&= \sum_{i=1}^{k-1} \beta^i \alpha(v_1, ..., v_{k-1}, v_i).
\end{aligned}$$

However, $\alpha(v_1, ..., v_{k-1}, v_i)$ is zero for $i = 1, ..., k-1$, because α is skew-symmetric. It follows that $\alpha(v_1, ..., v_k) = 0$.

Problem 3 (MTA 7.1-6) (PS 2009)

By analogy with the case for one forms, we define the wedge product $f_1 \wedge f_2 \wedge \cdots \wedge f_k$ by

$$\begin{aligned}
(f_1 \wedge f_2 \wedge \cdots \wedge f_k)(\alpha^1, \alpha^2, \ldots, \alpha^k) &= \sum_{\sigma \in S_k} (\operatorname{sign} \sigma)\, f_1(\alpha^{\sigma(1)}) f_2(\alpha^{\sigma(2)}) \ldots f_k(\alpha^{\sigma(k)}) \\
&= \sum_{\sigma \in S_k} (\operatorname{sign} \sigma)\, \alpha^{\sigma(1)}(f_1)\, \alpha^{\sigma(2)}(f_2) \ldots \alpha^{\sigma(k)}(f_k)
\end{aligned}$$

where $\alpha^1, \cdots, \alpha^k$ are arbitrary 1-forms. The notation above implicitly uses the canonical isomorphism $E^{**} \simeq E$.

If $\{e_1, \ldots, e_k\}$ and $\{f_1, \ldots, f_k\}$ span the same k-dimensional subspace, then we may express each vector f_i in the $\{e_1, \ldots, e_k\}$ basis:

$$f_i = a_{ji} e_j \qquad a_{ji} \in \mathbb{R}$$

. Then

$$\begin{aligned} f_1 \wedge \cdots \wedge f_k &= (a_{i_1,1}\, e_{i_1}) \wedge \cdots \wedge (a_{i_k,k}\, e_{i_k}) \\ &= a_{i_1,1} \ldots a_{i_k,k} (e_{i_1} \wedge \cdots \wedge e_{i_k}) && \text{(bilinearity)} \\ &= \sum_{\sigma \in S_k} a_{\sigma(1),1} \ldots a_{\sigma(k),k} (e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(k)}) && \text{(non-zero terms)} \\ &= \sum_{\sigma \in S_k} a_{\sigma(1),1} \ldots a_{\sigma(k),k} (\operatorname{sign} \sigma)(e_1 \wedge \cdots \wedge e_k) \\ &= \det(a_{ij})(e_1 \wedge \cdots \wedge e_k). \end{aligned}$$

Note that (a_{ij}) is just the matrix of the map ϕ mentioned in the question, with respect to the basis $\{e_1, \ldots, e_k\}$. In particular, $\det(a_{ij}) \neq 0$, since the sets $\{e_i\}$ and $\{f_j\}$ are linearly independent.

We now show the contrapositive to complete the argument. If $\{e_1, \ldots, e_k\}$ and $\{f_1, \ldots, f_k\}$ do not span the same k-dimensional subspace, then there exists $e_\alpha \in \{e_1, \ldots, e_k\}$ such that $e_\alpha \notin \operatorname{span}(f_1, \ldots, f_k)$, and hence $\{e_\alpha, f_1, \ldots, f_k\}$ is a linearly independent set. Now assume that we may write:

$$f_1 \wedge \cdots \wedge f_k = a\, e_1 \wedge \cdots \wedge e_k$$

Take the wedge product of both sides with e_α. The right hand side vanishes, and we are left with

$$e_\alpha \wedge f_1 \wedge \cdots \wedge f_k = 0.$$

But $\{e_\alpha, f_1, \ldots, f_k\}$ is a linearly independent set, so this is a contradiction (cf Corollary 7.1.10).

Problem 4 (MTA 7.2-2)

Suppose that $\dim E = \dim F = N$ and $\varphi^* \mu = \omega$. To prove that $\varphi \in L(E, F)$ is an isomorphism, we only need to show that φ is 1-1 (since $\dim E = \dim F$). We will accomplish this by a contradiction. Assume that φ is not 1-1, i.e., there is nonzero e_1 such that $\varphi(e_1) = 0$. Starting with e_1, find e_2, ..., e_N such that $\{e_1, ..., e_N\}$ are linearly independent. This can always be done. Then we get

$$\omega(e_1, ..., e_N) \neq 0$$

(since $\wedge^N E$ is 1-dimensional, so ω is a nonzero multiple of $e^1 \wedge e^2 \wedge \ldots \wedge e^N$, where $\{e^i\}$ is the dual basis to $\{e_i\}$). But on the other hand,

$$\varphi^*\mu(e_1, ..., e_N) = \mu(\varphi(e_1), ..., \varphi(e_N)) = 0,$$

which contradicts $\varphi^*\mu = \omega$. Therefore, we conclude that φ is 1-1, and it follows that φ is an isomorphism.

Problem 5 (MTA 7.3-2)

For the given φ, we have

$$D\varphi = \begin{bmatrix} 2x & 0 & 0 \\ 0 & z & y \end{bmatrix},$$

thus

$$\begin{aligned} \varphi^*(du) &= 2xdx, \\ \varphi^*(dv) &= zdy + ydz. \end{aligned}$$

Now we will compute $\alpha \wedge \beta$, $\varphi^*\alpha$, $\varphi^*\beta$ and $\varphi^*(\alpha \wedge \beta)$.

$$\begin{aligned} \alpha \wedge \beta &= v^2 du \wedge (uvdu \wedge dv) + dv \wedge (uvdu \wedge dv) \\ &= 0, \\ \varphi^*\alpha &= (yz)^2\varphi^*(du) + \varphi^*(dv) = (yz)^2 2xdx + zdy + ydz \\ &= 2xy^2z^2dx + zdy + ydz, \\ \varphi^*\beta &= x^2yz\varphi^*(du) \wedge \varphi^*(dv) = x^2yz(2xdx \wedge (zdy + ydz)) \\ &= 2x^3yz^2dx \wedge dy + 2x^3y^2zdx \wedge dz. \\ \varphi^*(\alpha \wedge \beta) &= \varphi^*(0) = 0. \end{aligned}$$