Chapter 4

Optimal Control in Lossy Networks

4.1 Introduction

Today, an increasingly number of applications demand remote control of plants over unreliable networks. The recent development of sensor web technology [68] enables the development of wireless sensor networks that can be immediately used for estimation and control of dynamical systems. In these systems issues of communication delay, data loss, and time-synchronization play critical roles. Communication and control become very tightly coupled and these two issues cannot be addressed independently in design and analysis of such systems. Consider, for example, the problem of navigating a fleet of vehicles using observations from a sensor web. Wireless nodes collect their sensor measurements and send them to a computing unit. This, in turn, generates state estimate for each vehicle and computes inputs, that are then delivered, using the same wireless channel, to the actuators onboard the vehicles. Due to the unreliability of the wireless channel, both observations underlying the estimate and control packets sent to each vehicle can be lost or delayed while travelling across the network. What is the amount of data loss that the control loop can tolerate to reliably perform the navigation task? Can communication protocols be designed to satisfy this constraint? The goal of this paper is to provide the first steps in answering such questions by examining the basic system-theoretic implications of using unreliable networks for control. This requires a generalization of classical control techniques that explicitly takes into account the stochastic nature of the communication channel.

Packet networks communication channels typically use one of two kinds of protocols: Transmission Control (TCP) or User Datagram (UDP). In the first case there is acknowledgement of received packets, while in the second case no confirmation feedback is provided on the commu-

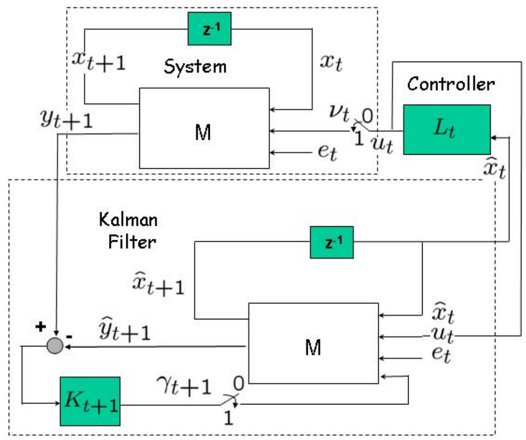

Figure 4.1: **Overview of the system.** We study the statistical convergence properties of the expected state covariance of the discrete time LQG control system, where both the observation and the control signal, transmitted over an unreliable communication channel, can be lost at each time step with probability $1-\bar{\gamma}$ and $1-\bar{\nu}$ respectively.

nication link. In this paper, we study the effect of data losses due to the unreliability of the network links under these two protocols. We generalize the Linear Quadratic Gaussian (LQG) optimal control problem to these problems by modeling the arrival of both observations and control packets as random processes whose parameters are related to the characteristics of the communication channel. Accordingly, two independent Bernoulli processes are considered, with parameters $\bar{\gamma}$ and $\bar{\nu}$, that govern packet losses between the sensors and the estimation-control unit, and between the latter and the actuation points (see Figure 4.1).

In our analyses, the distinction between the two classes of protocols resides exclusively in the availability of packet acknowledgements. Adopting the framework proposed by Imer *et al.* [32], we will refer therefore to TCP-like protocols if packet acknowledgements are available and to UDP-like protocols otherwise. We summarize our contributions as follows. For the TCP-like case the classic separation principle holds, and consequently the controller and estimator can be designed independently. Moreover, the optimal controller is a linear function of the state. In sharp contrast, for the UDP-like case, a counter-example demonstrates that the optimal controller is in general non-linear. In the special case when the state is fully observable and the observation noise is zero the optimal controller is indeed linear. We explicitly note that a similar, but slightly less general special case was previously analyzed in [32], where both observation and process noise are assumed to be zero and the input coefficient matrix to be invertible.

Our final set of results relate to convergence in the infinite horizon. Here, our previous results on estimation with missing observation packets [69] [45] are extended to the control case. We show the existence of a critical domain of values for the parameters of the Bernoulli arrival processes, $\bar{\nu}$ and $\bar{\gamma}$, outside which a transition to instability occurs and the optimal controller fails to stabilize the system. In particular, we show that under TCP-like protocols the critical arrival probabilities for the control and observation channel are independent of each other. This is another consequence of the fact that the separation principle holds for these protocols. In contrast, under UDP-like protocols the critical arrival probabilities for the control and observation channels are coupled. Here, the stability domain and performance of the optimal controller degrade considerably as compared with TCP-like protocols as shown in Figure 4.2.

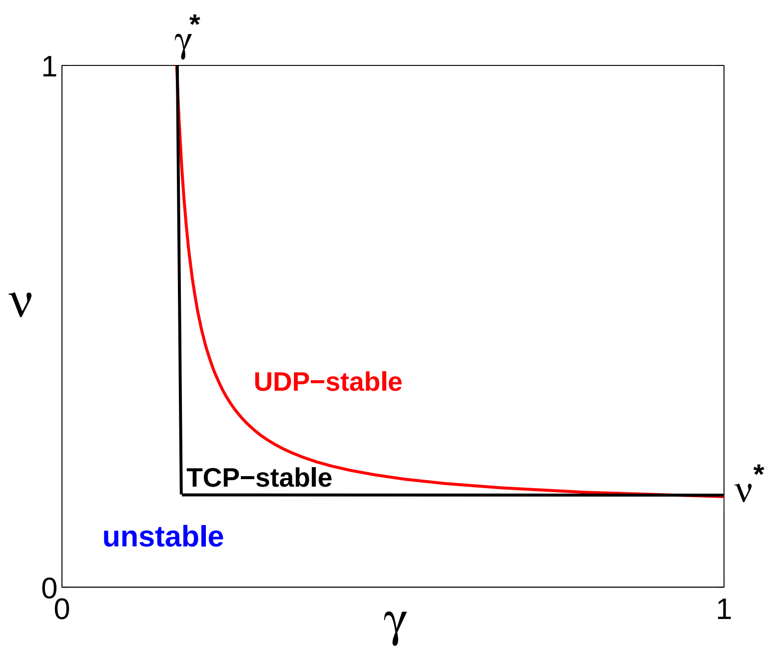

Figure 4.2: Region of stability for UDP-like and TCP-like optimal control relative to measurement packet arrival probability γ, and the control packet arrival probability ν.

Finally, we wish to mention some closely related research. Study of stability of dynamical systems where components are connected asynchronously via communication channels has received considerable attention in the past few years and our contribution can be put in the context of the previous literature. In [24] and [78], the authors proposed to place an estimator, i.e. a Kalman filter, at the sensor side of the link without assuming any statistical model for the data loss process. In [71], Smith *et al.* considered a suboptimal but computationally efficient estimator that can be applied when the arrival process is modeled as a Markov chain, which is more general than a Bernoulli process. Other work includes Nilsson *et al.* [57][59] who present the LQG optimal regulator with bounded delays between sensors and controller, and between the controller and the actuator. In this work, bounds for the critical probability values are not provided. Additionally, there is no analytic solution for the optimal controller. The case where dropped measurements are replaced by zeros is considered by Hadjicostis and Touri [27], but only in the scalar case. Other approaches include using the last received sample for control [59], or designing a dropout compensator [43], which combines estimation and control in a single process. However, the former approach does not consider optimal control and the latter is limited to scalar systems. Yu *et al.* [79] studied the design of an optimal controller with a single control channel and deterministic dropout rates. Seiler *et al.* [66] considered Bernoulli packet losses only between the plant and the controller and posed the controller design as an H_∞ optimization problem. Other authors [64] [14] [12] [75] model networked control systems with missing packets as Markovian jump linear systems (MJLSs), however this approach gives suboptimal controllers since the estimators are stationary. Finally, Elia [19][18] proposed to model the plant and the controller as deterministic time invariant discrete-time systems connected to zero-mean stochastic structured uncertainty. The variance of the stochastic perturbation is a function of the Bernoulli parameters, and the controller design is posed an an optimization problem to maximize mean-square stability of the closed loop system. This approach allows analysis of Multiple Input Multiple Output (MIMO) systems with many different controller and receiver compensation schemes [19], however, it does not include process and observation noise and the controller is restricted to be time-invariant, hence sub-optimal. There is also an extensive literature, inspired by Shannon's results on the maximum bit-rate that an imperfect channel can reliably carry, whose goal is to determine the minimum bit-rate that is needed to stabilize a system through feedback [77] [20] [28] [54] [72] [9] [47] [80] [42] [63]. This approach is somewhat different from ours since in a packet-based communication network, such as ATMs, Ethernet and Bluetooth, bits are grouped into packets and are considered as a single entity. Nonetheless there are several similarities that are not yet fully explored.

This paper considers the alternative approach where the external compensator feeding the controller is the optimal time varying Kalman gain. Moreover, this paper considers the general Multiple Input Multiple Output (MIMO) case, and gives some necessary and sufficient conditions for closed loop stability. The work of [32] is most closely related to this paper. However, we consider the more general case when the matrix C is not the identity and there is noise in the observation and in the process. In addition, we also give stronger necessary and sufficient conditions for existence of solution for the infinite horizon LQG.

The remainder of this paper is organized as follows. Section 2 provides a mathematical formulation of the problems we consider. Section 3 offers some preliminary results. Section 4 illustrates the TCP case, while the UDP case is studied in section 5. Finally, conclusions and directions for future work are offered in section 6.

4.2 Problem formulation

Consider the following linear stochastic system with intermittent observation and control packets:

$$x_{k+1} = Ax_k + Bu_k + w_k \quad (4.1)$$

$$u_k^a = \nu_k u_k^c \quad (4.2)$$

$$y_k = \gamma_k C x_k + v_k, \quad (4.3)$$

where u_k^a is the control input to the actuator, u_k^c is the desired control input computed by the controller, (x_0, w_k, v_k) are Gaussian, uncorrelated, white, with mean $(\bar{x}_0, 0, 0)$ and covariance (P_0, Q, R) respectively, and (γ_k, ν_k) are i.i.d. Bernoulli random variables with $P(\gamma_k = 1) = \bar{\gamma}$ and $P(\nu_k = 1) = \bar{\nu}$. The stochastic variable ν_k models the loss packets between the controller and the actuator: if the packet is correctly delivered then $u_k^a = u_k^c$, otherwise if it is lost then the actuator does nothing, i.e. $u_k^a = 0$. This compensation scheme is summarized by Equation (4.2). This modeling choice is not unique: for example if the control packet u_k^c is lost, then the actuator could use the previous control value, i.e. $u_k^a = u_{k-1}^a$. However, the latter control compensation is slightly more involved to analyze and it is left as future work. The stochastic variable γ_k models the packet loss between the sensor and the controller: if the packet is delivered then $y_k = Cx_k + v_k$, otherwise if it is lost then the controller reads pure noise, i.e. $y_k = v_k$. This observation model is summarized by Equation (4.3). A different observation formalism was proposed in [69], where the missing observation was modeled as an observation for which the measurement noise had infinite covariance. It is possible to show that both models are equivalent, but the one considered in this paper has the advantage to give rise to simpler analysis. This arises from the fact that when no packet is delivered, then the optimal estimator does not use the observation y_k at all, therefore its value is irrelevant.

Let us define the following information sets:

$$I_k = \begin{cases} F_k & \stackrel{\Delta}{=} & \{\mathbf{y}^k, \gamma^k, \nu^{k-1}\}, & \text{TCP-like} \\ G_k & \stackrel{\Delta}{=} & \{\mathbf{y}^k, \gamma^k\}, & \text{UDP-like} \end{cases} \tag{4.4}$$

where $\mathbf{y}^k = (y_k, y_{k-1}, \ldots, y_1)$, $\gamma^k = (\gamma_k, \gamma_{k-1}, \ldots, \gamma_1)$, and $\nu^k = (\nu_k, \nu_{k-1}, \ldots, \nu_1)$.

Consider also the following cost function:

$$J_N(\mathbf{u}^{N-1}, \bar{x}_0, P_0) = \mathbb{E}\left[x_N' W_N x_N + \sum_{k=0}^{N-1} (x_k' W_k x_k + \nu_k u_k' U_k u_k) \ \middle| \ \mathbf{u}^{N-1}, \bar{x}_0, P_0\right] \tag{4.5}$$

where $\mathbf{u}^{N-1} = (u_{N-1}, u_{N-2}, \ldots, u_1)$. Note that we are weighting the input only if it is successfully received at the plant. In fact, if it is not received, the plant applies zero input and therefore there is no energy expenditure.

We now look for a control input sequence $\mathbf{u}^{*N-1}$ as a function of the admissible information set I_k, i.e. $u_k = g_k(I_k)$, that minimizes the functional defined in Equation (4.5), i.e.

$$J_N^*(\bar{x}_0, P_0) \stackrel{\Delta}{=} \min_{\mathbf{u_k} = \mathbf{g}_k(I_k)} J_N(\mathbf{u}^{N-1}, \bar{x}_0, P_0), \tag{4.6}$$

where $I_k = \{F_k, G_k\}$ is one of the sets defined in Equation (4.4). The set F corresponds to the information provided under an acknowledgement-based communication protocols (TCP-like) in which successful or unsuccessful packet delivery at the receiver is acknowledged to the sender within the same sampling time period. The set G corresponds to the information available at the controller under communication protocols in which the sender receives no feedback about the delivery of the transmitted packet to the receiver (UDP-like). The UDP-like schemes are simpler to implement than the TCP-like schemes from a communication standpoint. Moreover UDP-like protocols includes broadcasting which you cannot do with TCP-like. However the price to pay is a less rich set of information. The goal of this paper is to design optimal LQG controllers and to estimate their performance for each of these classes of protocols for a general discrete-time linear stochastic system.

4.3 Mathematical Preliminaries

Before proceeding, let us define the following variables:

$$\begin{aligned} \hat{x}_{k|k} &\stackrel{\Delta}{=} \mathbb{E}[x_k \mid I_k], \\ e_{k|k} &\stackrel{\Delta}{=} x_k - \hat{x}_{k|k}, \\ P_{k|k} &\stackrel{\Delta}{=} \mathbb{E}[e_{k|k} e_{k|k}' \mid I_k]. \end{aligned} \tag{4.7}$$

Derivations below will make use of the following facts:

Lemma 4.3.1. *The following facts are true [70]:*

(a) $\mathbb{E}\left[(x_k-\hat{x}_k)\hat{x}_k' \mid I_k\right]=\mathbb{E}\left[e_{k|k}\hat{x}_k' \mid I_k\right]=0$

(b) $\mathbb{E}\left[x_k'Sx_k \mid I_k\right]=\hat{x}_k'S\hat{x}_k+\text{trace}\left(SP_{k|k}\right) \quad \forall S\geq 0$

(c) $\mathbb{E}\left[\mathbb{E}\left[\, g(x_{k+1}) \; |I_{k+1}\right] \mid I_k\right]=\mathbb{E}\left[g(x_{k+1}) \mid I_k\right], \forall g(\cdot).$

Proof. (a) It follows directly from the definition. In fact:

$$\begin{aligned}
\mathbb{E}\left[(x_k-\hat{x}_k)\hat{x}_k' \mid I_k\right] &= \mathbb{E}\left[x_k\hat{x}_k'-\hat{x}_k\hat{x}_k' \mid I_k\right] \\
&= \mathbb{E}\left[x_k \mid I_k\right]\hat{x}_k'-\hat{x}_k\hat{x}_k' \\
&= 0
\end{aligned}$$

(b) Using standard algebraic operations and the previous fact we have:

$$\begin{aligned}
\mathbb{E}\left[x_k'Sx_k \; |I_k\right] &= \mathbb{E}\left[(x_k-\hat{x}_k+\hat{x}_k)'S(x_k-\hat{x}_k+\hat{x}_k) \; |I_k\right] \\
&= \hat{x}_k'S\hat{x}_k+\mathbb{E}\left[(x_k-\hat{x}_k)'S(x_k-\hat{x}_k)\right]+2\mathbb{E}\left[\hat{x}_k'S(x_k-\hat{x}_k) \mid I_k\right] \\
&= \hat{x}_k'S\hat{x}_k+2\text{trace}(S\mathbb{E}[(x_k-\hat{x}_k)\hat{x}_k' \; |I_k])+\text{trace}(S\mathbb{E}[(x_k-\hat{x}_k)(x_k-\hat{x}_k)' \; |I_k]) \\
&= \hat{x}_k'S\hat{x}_k+\text{trace}\{SP_{k|k}\}
\end{aligned}$$

(c) Let $g()$ any measurable function, (X,Y,Z) be any random vectors, and p their probability distribution, then

$$\begin{aligned}
\mathbb{E}_{Y,Z}\left[g(X,Y,Z) \mid X\right] &= \int_Z\int_Y g(X,Y,Z)p(Y,Z|X)dYdZ \\
&= \int_Z\int_Y g(X,Y,Z)p(Y|Z,X)p(Z|X)dYdZ \\
&= \int_Z\left[\int_Y g(X,Y,Z)p(Y|Z,X)dY\right]p(Z|X)dZ \\
&= \mathbb{E}_Z\left[\,\mathbb{E}_Y\left[g(X,Y,Z) \mid Z,X\right] \mid X\right]
\end{aligned}$$

where we used the Bayes' Rule. Since by hypothesis $I_k \subseteq I_{k+1}$, then fact (c) follows from the above equality by substituting $I_k=X$ and $I_{k+1}=(X,Z)$. □

We now make the following computations that will be useful when deriving the equation for the optimal LQG controller.

$$\begin{aligned}
\mathbb{E}[x'_{k+1}Sx_{k+1} \mid I_k] &= \mathbb{E}[(Ax_k+\nu_kBu_k+w_k)'S(Ax_k+\nu_kBu_k+w_k) \mid I_k] \\
&= \mathbb{E}[x'_kA'SAx_k+\nu_k^2u'_kB'SBu_k+w'_kSw_k+2\nu_ku'_kB'SAx_k+2(Ax_k+\nu_kBu_k)w_k|I_k] \\
&= \mathbb{E}[x'_kA'SAx_k|F_k]+\bar{\nu}u'_kB'SBu_k+2\bar{\nu}u'_kB'SA\,\mathbb{E}[x_k|I_k]+\mathrm{trace}(S\mathbb{E}[w_kw'_k \mid F_k]) \\
&= \mathbb{E}[x'_kA'SAx_k \mid I_k]+\bar{\nu}u'_kB'SBu_k+\ 2\bar{\nu}u'_kB'SA\hat{x}_{k|k}+\mathrm{trace}(SQ) \qquad (4.8)
\end{aligned}$$

where both the independence of ν_k, w_k, x_k, and the zero-mean property of w_k are exploited. The previous expectation holds true for both the information sets, i.e. $I_k = F_k$ or $I_k = G_k$. Also

$$\begin{aligned}
\mathbb{E}[e'_{k|k}Te_{k|k} \mid I_k] &= \mathrm{trace}(T\mathbb{E}[e_{k|k}e'_{k|k} \mid I_k]) \\
&= \mathrm{trace}(TP_{k|k}), \quad \forall T \geq 0.
\end{aligned}$$

4.4 LQG control for TCP-like protocols

First, equations for the optimal estimator are derived. They will be needed to solve the LQG controller design problem, as it will be shown later.

4.4.1 Estimator Design

Equations for optimal estimator are derived using similar arguments used for the standard Kalman filtering equations. The innovation step is given by:

$$\begin{aligned}
\hat{x}_{k+1|k} &\triangleq \mathbb{E}[x_{k+1}|\nu_k, F_k] = \mathbb{E}[Ax_k+\nu_kBu_k+w_k|\nu_k, F_k] \\
&= A\mathbb{E}[x_k|F_k]+\nu_kBu_k = A\hat{x}_{k|k}+\nu_kBu_k \qquad (4.9) \\
e_{k+1|k} &\triangleq x_{k+1}-\hat{x}_{k+1|k} \\
&= Ax_k+\nu_kBu_k+w_k-(A\hat{x}+\nu_kBu_k) \\
&= Ae_{k|k}+w_k \qquad (4.10) \\
P_{k+1|k} &\triangleq \mathbb{E}[e_{k+1|k}e'_{k+1|k}\ |\nu_k, F_k] \\
&= \mathbb{E}\left[\left(Ae_{k|k}+w_k\right)\left(Ae_{k|k}+w_k\right)'\ |\nu_k, F_k\right] \\
&= A\mathbb{E}[e_{k|k}e'_{k|k}|F_k]A'+\mathbb{E}[w_kw'_k] \\
&= AP_{k|k}A'+Q, \qquad (4.11)
\end{aligned}$$

where the independence of w_k and F_k, and the requirement that u_k is a deterministic function of F_k, are used. Since y_{k+1}, γ_{k+1}, w_k and F_k are independent, the correction step is given by:

$$
\begin{aligned}
\hat{x}_{k+1|k+1} &= \hat{x}_{k+1|k} + \gamma_{k+1} K_{k+1} (y_{k+1} - C\hat{x}_{k+1|k}) && (4.12)\\
e_{k+1|k+1} &\stackrel{\Delta}{=} x_{k+1} - \hat{x}_{k+1|k+1} \\
&= x_{k+1} - \left(\hat{x}_{k+1|k} + \gamma_{k+1} K_{k+1} (Cx_{k+1} + v_{k+1} - C\hat{x}_{k+1|k})\right) \\
&= (I - \gamma_{k+1} K_{k+1} C) e_{k+1|k} - \gamma_{k+1} K_{k+1} v_{k+1} && (4.13)\\
P_{k+1|k+1} &= P_{k+1|k} - \gamma_{k+1} K_{k+1} C P_{k+1|k} \\
&= P_{k+1|k} - \gamma_{k+1} P_{k+1|k} C' (C P_{k+1|k} C' + R)^{-1} C P_{k+1|k} && (4.14)\\
K_{k+1} &\stackrel{\Delta}{=} P_{k+1|k} C' (C P_{k+1|k} C' + R)^{-1}, && (4.15)
\end{aligned}
$$

where we simply applied the standard derivation for the time varying Kalman filter using the following time varying system matrices: $A_k = A$, $C_k = \gamma_k C$, and $\mathrm{Cov}(v_k) = R$.

4.4.2 Controller design

Derivation of the optimal feedback control law and the corresponding value for the objective function will follow the dynamic programming approach based on the cost-to-go iterative procedure.

Define the optimal value function $V_k(x_k)$ as follows:

$$
\begin{aligned}
&V_N(x_N) \stackrel{\Delta}{=} \mathbb{E}[x_N' W_N x_N \mid F_N] \\
&V_k(x_k) \stackrel{\Delta}{=} \min_{u_k} \mathbb{E}[x_k' W_k x_k + \nu_k u_k' U_k u_k + V_{k+1}(x_{k+1}) | F_k].
\end{aligned} \qquad (4.16)
$$

where $k = N-1, \ldots, 1$. Using dynamic programming theory [7], one can show that $J_N^* = V_0(x_0)$. Under TCP-like protocols the following lemma holds true:

Lemma 4.4.1. *The value function $V_k(x_k)$ defined in Equations (4.16) for the system dynamics of Equations (4.1)-(4.1) under TCP-like protocols can be written as:*

$$
V_k(x_k) = \mathbb{E}[\, x_k' S_k x_k \mid F_k] + c_k, \quad k = N, \ldots, 0 \qquad (4.17)
$$

where the matrix S_k and the scalar c_k can be computed recursively as follows:

$$
\begin{aligned}
S_k &= A' S_{k+1} A + W_k - \bar{\nu} A' S_{k+1} B (B' S_{k+1} B + U_k)^{-1} B' S_{k+1} A && (4.18)\\
c_k &= \mathrm{trace}\left((A' S_{k+1} A + W_k - S_k) P_{k|k}\right) + \mathrm{trace}(S_{k+1} Q) + \mathbb{E}[c_{k+1} \mid F_k] && (4.19)
\end{aligned}
$$

with initial values $S_N = W_N$ *and* $c_N = 0$*. Moreover the optimal control input is given by:*

$$u_k = -(B'S_{k+1}B + U_k)^{-1}B'S_{k+1}A\hat{x}_{k|k} = L_k\hat{x}_{k|k}. \tag{4.20}$$

Proof. The proof follows an induction argument. The claim is certainly true for $k = N$ with the choice of parameters $S_N = W_N$ and $c_N = 0$. Suppose now that the claim is true for $k+1$, i.e. $V_{k+1}(x_{k+1}) = \mathbb{E}[\, x'_{k+1}S_{k+1}x_{k+1} \mid F_{k+1}] + c_{k+1}$. The value function at time step k is the following:

$$\begin{aligned} V_k(x_k) &= \min_{u_k} \mathbb{E}[x'_kW_kx_k + \nu_k u'_kU_ku_k + V_{k+1}(x_{k+1}) \mid F_k] \\ &= \min_{u_k} \mathbb{E}[x'_kW_kx_k + \nu_k u'_kU_ku_k + \mathbb{E}[x'_{k+1}S_{k+1}x_{k+1} + c_{k+1} \mid F_{k+1}] \mid F_k] \\ &= \min_{u_k} \mathbb{E}[x'_kW_kx_k + \nu_k u'_kU_ku_k + x'_{k+1}S_{k+1}x_{k+1} + c_{k+1} \mid F_k] \\ &= \mathbb{E}[x'_kW_kx_k + x'_kA'S_{k+1}Ax_k \mid F_k] + \mathrm{trace}(S_{k+1}Q) + \mathbb{E}[c_{k+1} \mid F_k] + \\ &\quad + \bar{\nu}\min_{u_k}\left(u'_k(U_k + B'S_{k+1}B)u_k + 2u'_kB'S_{k+1}A\hat{x}_{k|k}\right) \end{aligned} \tag{4.21}$$

where we used Lemma 1(c) to get the third equality, and Equation (4.8) to obtain the last equality. The value function is a quadratic function of the input, therefore the minimizer can be simply obtained by solving $\frac{\partial V_k}{\partial u_k} = 0$, which gives Equation (4.20). The optimal feedback is thus a simple linear function of the estimated state. If we substitute the minimizer back into Equation (4.21) we get:

$$\begin{aligned} V_k(x_k) &= \mathbb{E}[x'_kW_kx_k + x'_kA'S_{k+1}Ax_k \mid I_k] + \mathrm{trace}(S_{k+1}Q) + \mathbb{E}[c_{k+1} \mid I_k] - \\ &\quad - \bar{\nu}\hat{x}'_{k|k}A'S_{k+1}B(U_k + B'S_{k+1}B)^{-1}B'S_{k+1}A\hat{x}_{k|k} \\ &= \mathbb{E}[x'_kW_kx_k + x'_kA'S_{k+1}Ax_k - \bar{\nu}x'_kA'S_{k+1}B(U_k + B'S_{k+1}B)^{-1}B'S_{k+1}Ax_k \mid I_k] + \\ &\quad + \mathrm{trace}(S_{k+1}Q) + \mathbb{E}[c_{k+1} \mid I_k] + \bar{\nu}\,\mathrm{trace}(A'S_{k+1}B(U_k + B'S_{k+1}B)^{-1}B'S_{k+1}P_{k|k}) \end{aligned} \tag{4.22}$$

where we used Lemma 1(b). Therefore, the claim given by Equation (4.17) is satisfied also for time step k for all x_k if and only if the Equations (4.18) and (4.19) are satisfied. □

Since $J^*_N(\bar{x}_0, P_0) = V_0(x_0)$, from the lemma it follows that the cost function for the optimal LQG using TCP-like protocols is given by:

$$J^*_N = \bar{x}'_0S_0\bar{x}_0 + \mathrm{trace}(S_0P_0) + \sum_{k=0}^{N-1}\mathrm{trace}((A'S_{k+1}A + W_k - S_k)\mathbb{E}_\gamma[P_{k|k}] + S_{k+1}Q), \tag{4.23}$$

where we used the fact $\mathbb{E}[x_0' S_0 x_0] = \bar{x}_0' S_0 \bar{x}_0 + \text{trace}(S_0 P_0)$, and $\mathbb{E}_\gamma[\cdot]$ explicitly indicates that the expectation is calculated with respect to the arrival sequence $\{\gamma_k\}$.

It is important to remark that the error covariance matrices $\{P_{k|k}\}_{k=0}^N$ are stochastic since they depend on the sequence $\{\gamma_k\}$. Moreover, since the matrix $P_{k+1|k+1}$ is a nonlinear function of the previous time step matrix covariance $P_{k|k}$, as can be observed from Equations (4.11) and (4.15), the exact expected value of these matrices, $\mathbb{E}_\gamma[P_{k|k}]$, cannot be computed analytically, as shown in [69]. However, they can be bounded by computable deterministic quantities, as shown in [69] from which we can derive the following lemma:

Lemma 4.4.2 ([69]). *The expected error covariance matrix $\mathbb{E}_\gamma[P_{k|k}]$ satisfies the following bounds:*

$$\widetilde{P}_{k|k} \leq \mathbb{E}_\gamma[P_{k|k}] \leq \widehat{P}_{k|k} \qquad \forall k \geq 0, \tag{4.24}$$

where the matrices $\widehat{P}_{k|k}$ and $\widetilde{P}_{k|k}$ can be computed as follows:

$$\widehat{P}_{k+1|k} = A\widehat{P}_{k|k-1}A' + Q - \bar{\gamma}A\widehat{P}_{k|k-1}C'(C\widehat{P}_{k|k-1}C' + R)^{-1}C\widehat{P}_{k|k-1}A' \tag{4.25}$$

$$\widehat{P}_{k|k} = \widehat{P}_{k|k-1} - \bar{\gamma}\widehat{P}_{k|k-1}C'(C\widehat{P}_{k|k-1}C' + R)^{-1}C\widehat{P}_{k|k-1} \tag{4.26}$$

$$\widetilde{P}_{k+1|k} = (1-\bar{\gamma})A\widetilde{P}_{k|k-1}A' + Q \tag{4.27}$$

$$\widetilde{P}_{k|k} = (1-\bar{\gamma})\widetilde{P}_{k|k-1} \tag{4.28}$$

where the initial conditions are $\widehat{P}_{0|0} = \widetilde{P}_{0|0} = P_0$.

Proof. The proof is based on the observation that the matrices $P_{k+1|k}$ and $P_{k|k}$ are concave and monotonic functions of $P_{k|k-1}$. The proof is given in [69] and is thus omitted. □

From this lemma it follows that also the minimum achievable cost J_N^*, given by Equation (4.23), cannot be computed analytically, but can bounded as follows:

$$J_N^{min} \leq J_N^* \leq J_N^{max} \tag{4.29}$$

$$J_N^{max} = \bar{x}_0' S_0 \bar{x}_0 + \text{trace}(S_0 P_0) + \sum_{k=0}^{N-1} \text{trace}(S_{k+1}Q)) + \sum_{k=0}^{N-1} \text{trace}\left((A'S_{k+1}A + W_k - S_k)\widehat{P}_{k|k}\right) \tag{4.30}$$

$$J_N^{min} = \bar{x}_0' S_0 \bar{x}_0 + \text{trace}(S_0 P_0) + \sum_{k=0}^{N-1} \text{trace}(S_{k+1}Q) + \sum_{k=0}^{N-1} \text{trace}\left((A'S_{k+1}A + W_k - S_k)\widetilde{P}_{k|k}\right) \tag{4.31}$$

4.4.3 Finite and Infinite Horizon LQG control

The results derived in the previous sections can be summarized in the following theorem:

Theorem 4.4.1. *Consider the system (4.1)-(4.3) and consider the problem of minimizing the cost function (4.5) within the class of admissible policies $u_k = f(F_k)$, where F_k is the information available under TCP-like schemes, given in Equation (4.4). Then:*

(a) The separation principle still holds for TCP-like communication, since the optimal estimator, given by Equations (4.9),(4.11),(4.12),(4.14) and (4.15), is independent of the control input u_k.

(b) The optimal estimator gain K_k is time-varying and stochastic since it depends on the past observation arrival sequence $\{\gamma_j\}_{j=1}^{k}$.

(c) The optimal control input, given by Equations (4.20) and (4.18) with initial condition $S_N = W_N$, is a linear function of the estimated state $\hat{x}_{k|k}$, i.e. $u_k = L_k \hat{x}_{k|k}$, and is independent of the process sequences $\{\nu_k, \gamma_k\}$.

Proof. The proof follows from the results given in the previous sections. □

The infinite horizon LQG can be obtained by taking the limit for $N \to +\infty$ of the previous equations. However, as explained above, the matrices $\{P_{k|k}\}$ depend nonlinearly on the specific realization of the observation sequence $\{\gamma_k\}$, therefore the expected error covariance matrices $\mathbb{E}_\gamma[P_{k|k}]$ and the minimal cost J_N^* cannot be computed analytically and do not seem to have limit [69]. Differently from standard LQG optimal regulator [10], the estimator gain does not converge to a steady state value, but is strongly time-varying due to its dependence on the arrival process $\{\gamma_k\}$. Moreover, while the standard LQG optimal regulator always stabilizes the original system, in the case of observation and control packet losses, the stability can be lost if the arrival probabilities $\bar{\nu}, \bar{\gamma}$ are below a certain threshold. This observation come from the study of existence of solution for a Modified Riccati Algebraic Equation (MARE), $S = \Pi(S, A, B, W, U, \nu)$, which was introduced by [35] and studied in [37], [69] and [18], where the nonlinear operator $\Pi(\cdot)$ is defined as follows:

$$\Pi(S, A, B, Q, R, \nu) \stackrel{\Delta}{=} A'SA + W - \nu A'SB(B'SB + U)^{-1}B'SA \tag{4.32}$$

In particular, Equation (4.18), i.e. $S_{k+1} = \Pi(S_k, A, B, W, U, \nu)$, is the dual of the estimator equation presented in [69], i.e. $P_{k+1} = \Pi(P_k, A', C', Q, R, \gamma)$. The results about the MARE are summarized in the following lemma

Lemma 4.4.3. *Consider the modified Riccati equation defined in Equation (4.32). Let A be unstable, (A,B) be controllable, and $(A,W^{\frac{1}{2}})$ be observable. Then:*

(a) The MARE has a unique strictly positive definite solution S_∞ if and only if $\nu > \nu_c$, where ν_c is the critical arrival probability defined as:

$$\nu_c \stackrel{\Delta}{=} \inf_{\nu}\{0 \leq \nu \leq 1, S \geq 0) \,|\, S = \Pi(S,A,B,W,U,\nu)\}.$$

(b) The critical probability ν_c satisfy the following analytical bounds:

$$\begin{aligned} &p_{min} \leq \nu_c \leq p_{max} \\ &p_{min} \stackrel{\Delta}{=} 1 - \tfrac{1}{\max_i |\lambda_i^u(A)|^2} \\ &p_{max} \stackrel{\Delta}{=} 1 - \tfrac{1}{\prod_i |\lambda_i^u(A)|^2} \end{aligned}$$

where $\lambda_i^u(A)$ are the unstable eigenvalues of A. Moreover, $\nu_c = p_{min}$ when B is square and invertible, and $\nu_c = p_{max}$ when B is rank one.

(c) The critical probability can be numerically computed via the solution of the following quasi-convex LMIs optimization problem:

$$\begin{aligned} &\nu_c = \text{argmin}_{\bar{\nu}} \Psi_\nu(Y,Z) > 0, \quad 0 \leq Y \leq I. \\ &\Psi_\nu(Y,Z) = \begin{bmatrix} Y & \sqrt{\nu}(YA' + ZB') & \sqrt{1-\nu}YA' \\ \sqrt{\nu}(AY + BZ') & Y & 0 \\ \sqrt{1-\nu}AY & 0 & Y \\ & & \end{bmatrix} \end{aligned}$$

(d) If $\nu > \nu_c$, then $\lim_{k\to+\infty} S_k = S_\infty$ for all initial conditions $S_0 \geq 0$, where

$$S_{k+1} = \Pi(S_k,A,B,W,U,\nu)$$

.

Proof. The proof of facts (a),(c), and (d) can be found in [69]. The proof $\nu_c = p_{min}$ when B is square and invertible can be found in [35], and the proof $\nu_c = p_{max}$ when B is rank one in [18]. □

In [69] statistical analysis of the optimal estimator was given, which we report here for convenience:

Theorem 4.4.2 ([69]). *Consider the system (4.1)-(4.3) and the optimal estimator under TCP-like protocols, given by Equations (4.9),(4.11),(4.12),(4.14) and (4.15). Assume that* $(A, Q^{\frac{1}{2}})$ *is controllable,* (A,C) *is observable, and A is unstable. Then there exists a critical observation arrival probability* γ_c*, such that the expectation of estimator error covariance is bounded if and only if the observation arrival probability is greater than the critical arrival probability, i.e.*

$$\mathbb{E}_\gamma[P_{k|k}] \leq M \; \forall k \text{ iff } \bar{\gamma} > \gamma_c.$$

where M is a positive definite matrix possibly dependent on P_0*. Moreover, it is possible to compute a lower and an upper bound for the critical observation arrival probability* γ_c*, i.e.:*

$$p_{min} \leq \gamma_c \leq \gamma_{max} \leq p_{max}$$

, where:

$$\gamma_{max} \triangleq \inf_{\gamma}\{0 \leq \gamma \leq 1, P \geq 0) \,|\, P = \Pi(P, A', C', Q, R, \gamma)\},$$

where p_{min} *and* p_{max} *are defined in Lemma 4.4.3.*

Proof. The proof can be found in [69] and is therefore omitted. □

Using the previous theorem and the results from the previous section, we can prove the following theorem for the infinite horizon optimal LQG under TCP-like protocols:

Theorem 4.4.3. *Consider the same system as defined in the previous theorem with the following additional hypothesis:* $W_N = W_k = W$ *and* $U_k = U$*. Moreover, let* (A,B) *and* $(A, Q^{\frac{1}{2}})$ *be controllable, and let* (A,C) *and* $(A, W^{\frac{1}{2}})$ *be observable. Moreover, suppose that* $\bar{\nu} > \nu_c$ *and* $\bar{\gamma} > \gamma_{max}$*, where* ν_c *and* γ_{max} *are defined in Lemma 4.4.3 and in Theorem 4.4.2, respectively. Then we have:*

(a) The infinite horizon optimal controller gain is constant:

$$\lim_{k\to\infty} L_k = L_\infty = -(B'S_\infty B + U)^{-1} B' S_\infty A \tag{4.33}$$

(b) The infinite horizon optimal estimator gain K_k*, given by Equation (4.15), is stochastic and time-varying since it depends on the past observation arrival sequence* $\{\gamma_j\}_{j=1}^k$*.*

(c) The expected minimum cost can be bounded by two deterministic sequences:

$$\frac{1}{N} J_N^{min} \leq \frac{1}{N} J_N^* \leq \frac{1}{N} J_N^{max} \tag{4.34}$$

where J_N^{min}, J_N^{max} converge to the following values:

$$
\begin{aligned}
J_\infty^{max} &\triangleq \lim_{N\to+\infty}\frac{1}{N}J_N^{max}\\
&= \mathrm{trace}((A'S_\infty A+W-S_\infty)(\widehat{P}_\infty-\overline{\gamma}\widehat{P}_\infty C'(C\widehat{P}_\infty C'+R)^{-1}C\widehat{P}_\infty))+\mathrm{trace}(S_\infty Q)\\
J_\infty^{min} &\triangleq \lim_{N\to+\infty}\frac{1}{N}J_N^{min}\\
&= (1-\overline{\gamma})\mathrm{trace}\left((A'S_\infty A+W-S_\infty)\widetilde{P}_\infty\right)+\mathrm{trace}(S_\infty Q),
\end{aligned}
$$

and the matrices $S_\infty, \overline{P}_\infty, \underline{P}_\infty$ are the positive definite solutions of the following equations:

$$
\begin{aligned}
S_\infty &= A'S_\infty A+W-\overline{\nu}A'S_\infty B(B'S_\infty B+U)^{-1}B'S_\infty A\\
\overline{P}_\infty &= A\overline{P}_\infty A'+Q-\overline{\gamma}A\overline{P}_\infty C'(C\overline{P}_\infty C'+R)^{-1}C\overline{P}_\infty A'\\
\underline{P}_\infty &= (1-\overline{\gamma})A\underline{P}_\infty A'+Q
\end{aligned}
$$

Proof. (a) Since by hypothesis $\overline{\nu}>\nu_c$, from Lemma 4.4.3(d) follows that $\lim_{k\to+\infty}S_k=S_\infty$. Therefore Equation (4.33) follows from Equation (4.20).

(b) This follows from the dependence on the arrival sequence $\{\gamma_k\}$ of the optimal state estimator given by Equations (4.9),(4.11),(4.12),(4.14) and (4.15). Since $\overline{\nu}>\nu_c$

(c) Equation (4.25) can be written in terms of the MARE as $\widehat{P}_{k+1|k}=\Pi(\widehat{P}_{k|k-1},A',C',Q,R,\gamma)$, therefore since $\overline{\gamma}>\gamma_{max}$ from Lemma 4.4.3(d) it follows that $\lim_{k\to+\infty}\widehat{P}_{k|k-1}=\overline{P}_\infty$, where $\overline{P}_\infty$ is the solution of the MARE $\overline{P}_\infty=\Pi(\overline{P}_\infty,A',C',Q,R,\gamma)$. Also $\lim_{k\to+\infty}\widetilde{P}_{k|k-1}=\underline{P}_\infty$, where $\widetilde{P}_{k|k-1}$ is defined in Equation (4.27) and $\underline{P}_\infty$ is the solution of the Lyapunov equation $\widehat{P}_\infty=\tilde{A}\widehat{P}_\infty\tilde{A}'+Q$, where $\tilde{A}=\sqrt{1-\overline{\gamma}}A$. Such solution clearly exists since $\sqrt{1-\gamma}<\frac{1}{p_{min}}=\frac{1}{\max_i|\lambda_i^u(A)|}$ and thus the matrix $\tilde{A}$ is strictly stable. From Equations (4.26) and (4.28) it follows that $\lim_{k\to+\infty}\widehat{P}_{k|k}=\overline{P}_\infty-\overline{\gamma}\overline{P}_\infty C'(C\overline{P}_\infty C'+R)^{-1}C\overline{P}_\infty$ and $\lim_{k\to+\infty}\widetilde{P}_{k|k}=(1-\overline{\gamma})\underline{P}_\infty$. Also $\lim_{k\to+\infty}S_{k+1}=\lim_{k\to+\infty}S_k=S_\infty$. Finally from Equations (4.29) - (4.31) and the previous observations follow the claim.

□

4.5 LQG control for UDP-like protocols

In this section equations for the optimal estimator and controller design for the case of communication protocols that do not provide any kind of acknowledgment of successful packet delivery (UDP-like). This case corresponds to the information set G_k, as defined in Equation (4.4). Some of the derivations are analogous to the previous section and are therefore skipped.

4.5.1 Estimator Design

We derive the equations for the optimal estimator using similar arguments to the standard Kalman filtering equations. The innovation step is given by:

$$\begin{aligned}
\hat{x}_{k+1|k} &\stackrel{\Delta}{=} \mathbb{E}[x_{k+1}|G_k] = \mathbb{E}[Ax_k + \nu_k Bu_k + w_k|G_k] \\
&= A\mathbb{E}[x_k|G_k] + \mathbb{E}[\nu_k]Bu_k \\
&= A\hat{x}_{k|k} + \bar{\nu}Bu_k \qquad (4.35) \\
e_{k+1|k} &\stackrel{\Delta}{=} x_{k+1} - \hat{x}_{k+1|k} \\
&= Ax_k + \nu_k Bu_k + w_k - (A\hat{x}_{k|k} + \bar{\nu}Bu_k) \\
&= Ae_{k|k} + (\nu_k - \nu)Bu_k + w_k \qquad (4.36) \\
P_{k+1|k} &\stackrel{\Delta}{=} \mathbb{E}[e_{k+1|k}e'_{k+1|k}\,|G_k] \\
&= A\mathbb{E}[e_{k|k}e'_{k|k}|G_k]A' + \mathbb{E}[(\nu_k - \nu)^2]Bu_k u'_k B' + \mathbb{E}[w_k w'_k] \\
&= AP_{k|k}A' + \bar{\nu}(1-\bar{\nu})Bu_k u'_k B' + Q, \qquad (4.37)
\end{aligned}$$

where we used the independence and zero-mean of w_k, $(\nu_k - \bar{\nu})$, and G_k, and the fact that u_k is a deterministic function of the information set G_k. Note how under UDP-like communication, differently from TCP-like, the error covariance $P_{k+1|k}$ depends explicitly on the control input u_k. This is the main difference with control feedback systems under TCP-like protocols.

The correction step is the same as for the TCP case:

$$\begin{aligned}
\hat{x}_{k+1|k+1} &= \hat{x}_{k+1|k} + \gamma_{k+1}K_{k+1}(y_{k+1} - C\hat{x}_{k+1|k}) \\
P_{k+1|k+1} &= P_{k+1|k} - \gamma_{k+1}K_{k+1}CP_{k+1|k}, \qquad (4.38) \\
K_{k+1} &\stackrel{\Delta}{=} P_{k+1|k}C'(CP_{k+1|k}C' + R)^{-1}, \qquad (4.39)
\end{aligned}$$

where again we considered a time varying system with $A_k = A$ and $C_k = \gamma_k C$ as we did for the optimal estimator under TCP-like protocols.

4.5.2 Controller design: General case

In this section, we show that the optimal LQG controller, under UDP-like communication protocols, is in general not a linear function of the state estimate, and that the estimator and controller design cannot be separated anymore. To show this, we construct a counter-example considering a simple scalar system and we proceed using the dynamic programming approach. Let us consider the scalar system where $A=1, B=1, C=1, W_N=W_k=1, U_k=0, R=1, Q=0$. Similarly to the TCP case, we define the value function, $V_k(x_k)$, as in Equations (4.16) where we just need to substitute the information set F_k with G_k. For $k=N$, the value function is given by $V_N(x_N)=\mathbb{E}[x_N' W_N x_N \mid G_N]=\mathbb{E}[x_N^2 \mid G_N]$. For $k=N-1$ we have:

$$\begin{aligned}
V_{N-1}(x_{N-1}) &= \min_{u_{N-1}} \mathbb{E}[x_{N-1}^2 + V_N(x_N) \mid G_{N-1}] \\
&= \min_{u_{N-1}} \mathbb{E}[x_{N-1}^2 + x_N^2 \mid G_{N-1}] \\
&= \min_{u_{N-1}} \mathbb{E}[x_{N-1}^2 + (x_{N-1}+\nu_{N-1}u_{N-1})^2 \mid G_{N-1}] \\
&= \min_{u_{N-1}} (\mathbb{E}[2x_{N-1}^2|G_{N-1}] + \mathbb{E}[\nu_{N-1}^2]u_{N-1}^2 + 2u_{N-1}\mathbb{E}[\nu_{N-1}]\mathbb{E}[x_{N-1}|G_{N-1}]) \\
&= \min_{u_{N-1}} (\mathbb{E}[2x_{N-1}^2|G_{N-1}] + \bar{\nu}u_{N-1}^2 + 2\bar{\nu}u_{N-1}\hat{x}_{N-1|N-1}),
\end{aligned}$$

where we used the independence of ν_{N-1} and G_{N-1}, and the fact that u_{N-1} is a deterministic function of the information set G_{N-1}. The cost is a quadratic function of the input u_{N-1}, therefore the minimizer can be simply obtained by finding $\frac{\partial V_{N-1}}{\partial u_{N-1}}=0$, which is given by $u_{N-1}^*=-\hat{x}_{N-1|N-1}$. If we substitute back u_{N-1}^* into the value function we have:

$$\begin{aligned}
V_{N-1}(x_{N-1}) = & \ \mathbb{E}[2x_{N-1}^2|G_{N-1}] - \bar{\nu}\hat{x}_{N-1|N-1}^2 \\
= & \ \mathbb{E}[(2-\bar{\nu})x_{N-1}^2|G_{N-1}] + \bar{\nu}P_{N-1|N-1}
\end{aligned}$$

where we used Lemma 4.3.1(b). Before proceeding note that:

$$\begin{aligned}
P_{N-1|N-1} &= P_{N-1|N-2} - \gamma_{N-1}\frac{P_{N-1|N-2}^2}{P_{N-1|N-2}+1} \\
&= P_{N-1|N-2} - \gamma_{N-1}\left(P_{N-1|N-2} - 1 + \frac{1}{P_{N-1|N-2}+1}\right) \\
&= (1-\gamma_{N-1})\left(P_{N-2|N-2} + \bar{\nu}(1-\bar{\nu})u_{N-2}^2\right) + \gamma_{N-1} + \\
&\quad +\gamma_{N-1}\frac{1}{P_{N-2|N-2}+\bar{\nu}(1-\bar{\nu})u_{N-2}^2+1} \\
\mathbb{E}[P_{N-1|N-1}|G_{N-2}] &= (1-\bar{\gamma})\left(P_{N-2|N-2}+\bar{\nu}(1-\bar{\nu})u_{N-2}^2\right) + \bar{\gamma} + \bar{\gamma}\frac{1}{P_{N-2|N-2}+\bar{\nu}(1-\bar{\nu})u_{N-2}^2+1} \\
\mathbb{E}[x_{N-1}^2|G_{N-2}] &= \mathbb{E}[(x_{N-2}+\nu_{N-2}u_{N-2})^2|G_{N-2}] \\
&= \mathbb{E}[x_{N-2}^2|G_{N-2}] + 2\mathbb{E}[\nu_{N-2}]\mathbb{E}[x_{N-2}|G_{N-2}]u_{N-2} + \mathbb{E}[\nu_{N-2}]u_{N-2}^2 \\
&= \mathbb{E}[x_{N-2}^2|G_{N-2}] + 2\bar{\nu}\hat{x}_{N-2|N-2}u_{N-2} + \bar{\nu}u_{N-2}^2,
\end{aligned}$$

where we used Equations (4.37)-(4.39), and the fact that u_{N-2} and $P_{N-2|N-2}$ are a deterministic function of the information set G_{N-2}. Using the previous equations we proceed to compute the value function for $k = N-2$:

$$\begin{aligned} V_{N-2}(x_{N-2}) &= \min_{u_{N-2}} \mathbb{E}[x_{N-2}^2 + V_{N-1}(x_{N-1}) \mid G_{N-2}] \\ &= \min_{u_{N-2}} \mathbb{E}[x_{N-2}^2 + (2-\overline{\nu})x_{N-1}^2 + \overline{\nu}P_{N-1|N-1} \mid G_{N-2}] \\ &= (3-\overline{\nu})\mathbb{E}[x_{N-2}^2 | G_{N-2}] + \overline{\nu}(1-\overline{\gamma})P_{N-2|N-2} + \overline{\nu}\overline{\gamma} + \\ &\quad + \min_{u_{N-1}} \Big(2\overline{\nu}(2-\overline{\nu})\hat{x}_{N-2|N-2}u_{N-2} + \overline{\nu}(2-\overline{\nu})u_{N-2}^2 + \\ &\quad + \overline{\nu}^2(1-\overline{\nu})(1-\overline{\gamma})u_{N-2}^2 + \overline{\nu}\overline{\gamma}\frac{1}{P_{N-2|N-2} + \overline{\nu}(1-\overline{\nu})u_{N-2}^2 + 1} \Big) \end{aligned}$$

The first three terms inside the round parenthesis are convex quadratic functions of the control input u_{N-2}, however the last term is not. Therefore, the minimizer u_{N-2}^* is, in general, a non-linear function of the information set G_k. The nonlinearity of the optimal controller arises from the fact that the correction error covariance matrix $P_{k+1|k+1}$ is a non-linear function of the innovation error covariance $P_{k+1|k}$, as it can be seen in Equations (4.38) and (4.39). The only case when $P_{k+1|k+1}$ is linear in $P_{k+1|k}$ is when measurement noise covariance $R = 0$ and the observation matrix C is square and invertible, from which follows that the optimal control is linear in the estimated states. However it is important to remark that the separation principle still does not hold even for this special case, since the control input affects the estimator error covariance.

We can summarize these results in the following theorem:

Theorem 4.5.1. *Let us consider the stochastic system defined in Equations (4.1) with horizon $N \geq 2$. Then:*

(a) The separation principle does not hold since the estimator error covariance depends on the control input, as shown in Equation (4.37).

(b) The optimal control feedback $u_k = g_k^*(G_k)$ that minimizes the cost functional defined in Equation (4.5) under UDP-like protocols is, in general, a nonlinear function of information set G_k.

(c) The optimal control feedback $u_k = g_k^*(G_k)$ is a linear function of the estimated state $\hat{x}_{k|k}$ if and only if the matrix C is invertible and there is no measurement noise.

The next section will compute explicitly the optimal control for the special case and will give necessary and sufficient conditions for stability and performance of the infinite horizon scenario.

4.5.3 Special Case: R=0 and C invertible

Without loss of generality we can assume $C = I$, since the linear transformation $z = Cx$ would give an equivalent system where the matrix C is the indentity. Let us now consider the case when there is no measurement noise, i.e. $R = 0$. These assumption mean that it is possible to measure the state x_k when a packet is delivered. In this case the estimator Equations (4.37)-(4.39) simplify as follows:

$$
\begin{aligned}
K_{k+1} &= I && (4.40)\\
P_{k+1|k+1} &= (1-\gamma_{k+1})P_{k+1|k} \\
&= (1-\gamma_{k+1})(A'P_{k|k}A + Q + \bar{\nu}(1-\bar{\nu})Bu_k u_k' B') && (4.41)\\
\mathbb{E}[P_{k+1|k+1}|G_k] &= (1-\bar{\gamma})(A'P_{k|k}A + Q + \bar{\nu}(1-\bar{\nu})Bu_k u_k' B') && (4.42)
\end{aligned}
$$

where in the last equation we used independence of γ_{k+1} and G_k, and we used the fact that $P_{k|k}$ is a deterministic function of G_k.

Similarly to what done in the analysis of TCP-like optimal control, we claim that the value function $V_k^*(x_k)$ can be written as follows:

$$
V_k(x_k) = \hat{x}_{k|k}' S_k \hat{x}_{k|k} + \text{trace}(T_k P_{k|k}) + \text{trace}(D_k Q) \qquad (4.43)
$$

for $k = N, \ldots, 0$. This is clearly true for $k = N$, in fact we have:

$$
V_N(x_N) = \mathbb{E}[x_N' W_N x_N | G_N] = \hat{x}_{N|N}' W_N \hat{x}_{N|N} + \text{trace}(W_N P_{N|N})
$$

where we used Lemma 4.3.1(b), therefore the statement is satisfied by $S_N = W_N, T_N = W_N, D_N = 0$. Note that Equation (4.43) can be rewritten as follows:

$$
V_k(x_k) = \mathbb{E}[x_k' S_k x_k | G_k] + \text{trace}\big((T_k - S_k)P_{k|k}\big) + \text{trace}(D_k Q)
$$

where we used once again Lemma 4.3.1(b). Moreover, to simplify notation we define $H_k \overset{\Delta}{=} (T_k - S_k)$.

Let us suppose that Equation (4.43) is true for $k+1$ and let us show by induction it holds true for k:

$$\begin{aligned}
V_k(x_k) &= \min_{u_k} \mathbb{E}[x_k' W_k x_k + \nu_k u_k' U_k u_k + V_{k+1}(x_{k+1}) \mid G_k] \\
&= \min_{u_k} \left(\mathbb{E}[x_k' W_k x_k + \nu_k u_k' U_k u_k + x_{k+1}' S_{k+1} x_{k+1} + \mathrm{trace}(H_{k+1} P_{k+1|k+1}) + \mathrm{trace}(D_{k+1} Q) \mid G_k]\right) \\
&= \mathbb{E}[x_k'(W_k + A'S_{k+1}A)x_k | G_k] + \mathrm{trace}(S_{k+1}Q) + (1-\overline{\gamma})\mathrm{trace}(H_{k+1}(A'P_{k|k}A + Q)) + \mathrm{trace}(D_{k+1}Q) + \\
&\quad + \min_{u_k}\left(\overline{\nu} u_k' U_k u_k + \overline{\nu} u_k' B' S_{k+1} B u_k + 2\overline{\nu} u_k' B' S_{k+1} A \hat{x}_{k|k} + \overline{\nu}(1-\overline{\nu})(1-\overline{\gamma})\mathrm{trace}(H_{k+1} B u_k u_k' B')\right) \\
&= \mathbb{E}[x_k'(W_k + A'S_{k+1}A)x_k | G_k] + \mathrm{trace}\big((D_{k+1} + (1-\overline{\gamma})H_{k+1})Q\big) + (1-\overline{\gamma})\mathrm{trace}(AH_{k+1}A'P_{k|k}) + \\
&\quad + \mathrm{trace}(S_{k+1}Q) + \overline{\nu} \min_{u_k}\left(u_k'\big(U_k + B'(S_{k+1} + (1-\overline{\nu})(1-\overline{\gamma})H_{k+1})B\big)u_k + 2u_k' B' S_{k+1} A \hat{x}_{k|k}\right) \\
&= \hat{x}_{k|k}'(W_k + A'S_{k+1}A)\hat{x}_{k|k} + \mathrm{trace}\big((D_{k+1} + (1-\overline{\gamma})T_{k+1} + \overline{\gamma}S_{k+1})Q\big) + \\
&\quad + \mathrm{trace}\Big((W_k + \overline{\gamma}A'S_{k+1}A + (1-\overline{\gamma})AT_{k+1}A')P_{k|k}\Big) + \\
&\quad + \overline{\nu} \min_{u_k}\left(u_k'\big(U_k + B'((1-\overline{\alpha})S_{k+1} + \overline{\alpha}T_{k+1})B\big)u_k + 2u_k' B' S_{k+1} A \hat{x}_{k|k}\right),
\end{aligned}$$

where we defined $\overline{\alpha} = (1-\overline{\nu})(1-\overline{\gamma})$, we used Lemma 4.3.1(c) to get the second equality, and Equations (4.8) and (4.42) to get the last equality. Since the quantity inside the big round parenthesis a convex quadratic function, the minimizer is the solution of $\frac{\partial V_k}{\partial u_k} = 0$ which is given by:

$$u_k^* = -\Big(U_k + B'\big((1-\overline{\alpha})S_{k+1} + \overline{\alpha}T_{k+1}\big)B\Big)^{-1} B' S_{k+1} A \hat{x}_{k|k} \tag{4.44}$$

$$= L_k \hat{x}_{k|k} \tag{4.45}$$

which is linear function of the estimated state $\hat{x}_{k|k}$. Substituting back into the value function we get:

$$\begin{aligned}
V_k(x_k) &= \hat{x}_{k|k}'(W_k + A'S_{k+1}A)\hat{x}_{k|k} + \mathrm{trace}\big((D_{k+1} + (1-\overline{\gamma})T_{k+1} + \overline{\gamma}S_{k+1})Q\big) + \\
&\quad + \mathrm{trace}\Big((W_k + A'S_{k+1}A + (1-\overline{\gamma})AT_{k+1}A')P_{k|k}\Big) - \overline{\nu}\hat{x}_{k|k}' A' S_{k+1} B L_k \hat{x}_{k|k} \\
&= \hat{x}_{k|k}'(W_k + \overline{\gamma}A'S_{k+1}A - \overline{\nu}\hat{x}_{k|k}' A' S_{k+1} B L_k)\hat{x}_{k|k} + \mathrm{trace}\big((D_{k+1} + (1-\overline{\gamma})T_{k+1} + \overline{\gamma}S_{k+1})Q\big) + \\
&\quad + \mathrm{trace}\Big((W_k + A'S_{k+1}A + (1-\overline{\gamma})AT_{k+1}A')P_{k|k}\Big),
\end{aligned}$$

where we used Lemma 4.3.1(b) in the last equality. From the last equation we see that the value function can be written as in Equation (4.43) if and only if the following equations are satisfied:

$$\begin{aligned}
S_k &= A'S_{k+1}A + W_k - \overline{\nu}A'S_{k+1}B\left(U_k + B'\left((1-\overline{\alpha})S_{k+1} + \overline{\alpha}T_{k+1}\right)B\right)^{-1} B'S_{k+1}A \\
&= \Phi^S_{\gamma,\nu}(S_{k+1}, T_{k+1}) && (4.46) \\
T_k &= (1-\overline{\gamma})A'T_{k+1}A + \overline{\gamma}A'S_{k+1}A + W_k \\
&= \Phi^T_{\gamma,\nu}(S_{k+1}, T_{k+1}) && (4.47) \\
D_k &= (1-\overline{\gamma})T_{k+1} + \overline{\gamma}S_{k+1} + D_{k+1} && (4.48)
\end{aligned}$$

The optimal minimal cost for the finite horizon, $J_N^* = V_0(x_0)$ is then given by:

$$J_N^* = \overline{x}_0' S_0 \overline{x}_0 + \text{trace}(S_0 P_0) + \sum_{k=1}^{N} \text{trace}\Big(\big((1-\overline{\gamma})T_k + \overline{\gamma} S_k\big) Q \Big) \tag{4.49}$$

For the infinite horizon optimal controller, necessary and sufficient condition for the average minimal cost $J_\infty \stackrel{\Delta}{=} \lim_{N\to+\infty} \frac{1}{N} J_N^*$ to be finite is that the coupled iterative Equations (4.46) and (4.47) should converge to a finite value S_∞ and T_∞ as $N \to +\infty$. In the work of Imer *et al.* [32] similar equations were derived for the optimal LQG control under UDP for the same framework with the additional conditions $Q = 0$ and B square and invertible. They find necessary and sufficient conditions for those equations to converge. Unfortunately, these conditions do not hold for the general case when B in not square. This is a very frequent situation in control systems, where in general we simply have (A,B) controllable.

Theorem 4.5.2. *Also, assume that the pair $(A, W^{1/2})$ is observable. Consider the following operator:*

$$\Upsilon(S,T,L) = A'SA + W + 2\overline{\nu} A'SBL + \overline{\nu} L' \Big(U + B' \big((1-\overline{\alpha})S + \overline{\alpha} T \big) B \Big) L \tag{4.50}$$

Then the following claims are equivalent:

(a) There exist a matrix $\tilde{L}$ and positive definite matrices $\tilde{S}$ and $\tilde{T}$ such that:

$$\tilde{S} > 0,\ \tilde{T} > 0,\ \tilde{S} = \Upsilon(\tilde{S},\tilde{T},\tilde{L}),\ \tilde{T} = \Phi^T(\tilde{S},\tilde{T})$$

(b) Consider the sequences:

$$S_{k+1} = \Phi^S(S_k,T_k),\ \ T_{k+1} = \Phi^T(S_k,T_k)$$

where the operators $\Phi^S(\cdot), \Phi^T(\cdot)$ are defined in Equations (4.46) and (4.47). For any initial condition $S_0, T_0 \geq 0$ we have

$$\lim_{k\to\infty} S_k = S_\infty,\ \ \lim_{k\to\infty} T_k = T_\infty$$

and S_∞, T_∞ are the unique positive definite solution of the following equations

$$S_\infty > 0,\ T_\infty > 0,\ S_\infty = \Phi^S(S_\infty,T_\infty),\ \ T_\infty = \Phi^T(S_\infty,T_\infty)$$

The convergence of Equations (4.46) and (4.47) depend on the control and observation arrival probabilities $\overline{\gamma}, \overline{\nu}$. General analytical conditions for convergence are not available, but some necessary and sufficient conditions can be found.

Lemma 4.5.1. *Let us consider the fixed points of Equations (4.46) and (4.47), i.e.* $S = \Phi^S(S,T), T = \Phi^T(S,T)$ *where* $S,T \geq 0$. *Let A be unstable. A necessary condition for existence of solution is*

$$|A|^2(\bar{\gamma} + \bar{\nu} - 2\overline{\gamma\nu}) < \bar{\gamma} + \bar{\nu} - \overline{\gamma\nu} \tag{4.51}$$

where $|A| \triangleq \max_i |\lambda_i(A)|$ *is the largest eigenvalue of the matrix A.*

Lemma 4.5.2. *Let us consider the fixed points of Equations (4.46) and (4.47), i.e.* $S = \Phi^S(S,T), T = \Phi^T(S,T)$ *where* $S,T \geq 0$. *Let A be unstable,* $(A, W^{1/2})$ *observable and B square and invertible. Then a sufficient condition for existence of solution is*

$$|A|^2(\bar{\gamma} + \bar{\nu} - 2\overline{\gamma\nu}) < \bar{\gamma} + \bar{\nu} - \overline{\gamma\nu} \tag{4.52}$$

where $|A| \triangleq \max_i |\lambda_i(A)|$ *is the largest eigenvalue of the matrix A.*

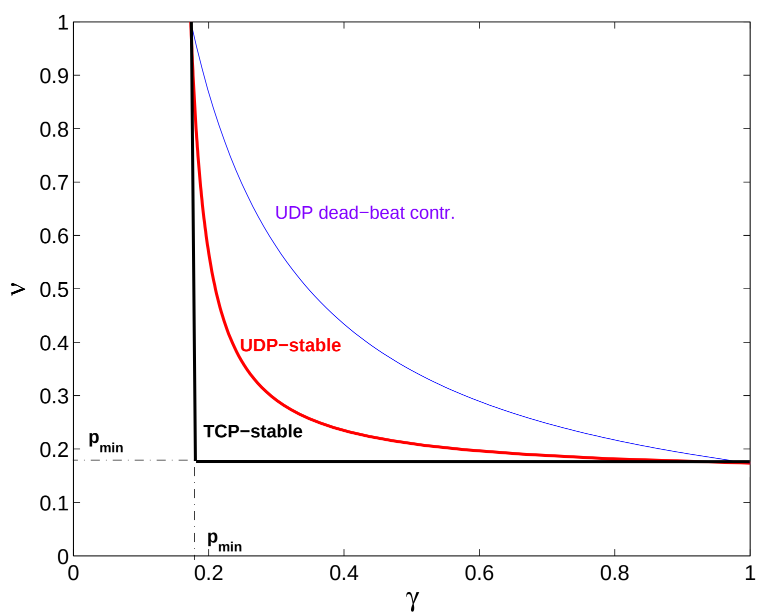

Figure 4.3: Region of convergence for UDP-like and TCP-like optimal control in the scalar case. These bounds are tight in the scalar case. The thin solid line corresponds to the boundary of the stability region for a dead-beat controller under UDP-like protocols as given by [32], which is much more restrictive than what can be achieved with optimal UDP controllers.

A graphical representation of the stability bounds are shown in Figure 4.3, where we considered a scalar system with parameters $|A| = 1.1$, which gives the critical probability $p_{min} = 1 - 1/|A|^2 = 1.173$ as defined in Theorem 4.4.2. The critical arrival probabilities for TCP-like optimal control are $\gamma_c = \nu_c = p_{min}$. The boundary for the stability region of optimal control under UDP-like protocols given in Lemma 4.5.2 can be written also as $\bar{\nu} > \frac{\bar{\gamma}(A^2-1)}{\bar{\gamma}(2A^2-1)+1-A^2}$ for $\bar{\gamma} > p_{min}$. It is important to remark that the stability region of optimal control under UDP-like protocols is larger than the stability region obtained using a dead-beat controller proposed in [32], i.e. $u_k = -\gamma_k B^{-1}Ay_k = -\gamma_k B^{-1}Ax_k$, which is given by $\overline{\gamma\nu} > 1-1/|A|^2$ and graphically shown in Figure 4.3 . This is not surprising since the dead-beat controller is rather aggressive and requires a large gain L, which increases the estimator error covariance in Equation (4.42). Indeed, as shown in the constructive proof of Lemma 4.5.2, controllers with similar structure but smaller gains, i.e. $u_k = -\eta\gamma_k B^{-1}Ay_k = -\eta\gamma_k B^{-1}Ax_k$ where $\eta < 1$, have a larger region of stability.

We can summarize the results of this section in the following theorem

Theorem 4.5.3. *Consider the system (4.1)-(4.3) and consider the problem of minimizing the cost function (4.5) within the class of admissible policies $u_k = f(G_k)$, where G_k is the information available under TCP-like schemes, given in Equation (4.4). Assume also that $R = 0$ and C is square and invertible. Then:*

(a) The optimal estimator gain is constant and in particular $K_k = I$ if $C = I$.

(b) The infinite horizon optimal control exists if and only if there exists positive definite matrices $S_\infty, T_\infty > 0$ such that $S_\infty = \Phi^S(S_\infty, T_\infty)$ and $T_\infty = \Phi^T(S_\infty, T_\infty)$, where Φ^S and Φ^S are defined in Equations (4.46) and (4.47).

(c) The infinite horizon optimal controller gain is constant:

$$\lim_{k\to\infty} L_k = L_\infty = -(B'(\bar{\alpha}T_\infty + (1-\bar{\alpha})S_\infty)B + U)^{-1}B'S_\infty A \tag{4.53}$$

(d) A necessary condition for existence of $S_\infty, T_\infty > 0$ is

$$|A|^2(\bar{\gamma} + \bar{\nu} - 2\overline{\gamma\nu}) < \bar{\gamma} + \bar{\nu} - \overline{\gamma\nu} \tag{4.54}$$

where $|A| \stackrel{\Delta}{=} \max_i |\lambda_i(A)|$ is the largest eigenvalue of the matrix A. This condition is also sufficient if B is square and invertible.

(e) The expected minimum cost converges:

$$J_\infty^* = \lim_{k\to\infty} \frac{1}{N} J_N^* = \mathrm{trace}\big((1-\bar{\gamma})T_\infty + \bar{\gamma}S_\infty)Q\big) \tag{4.55}$$

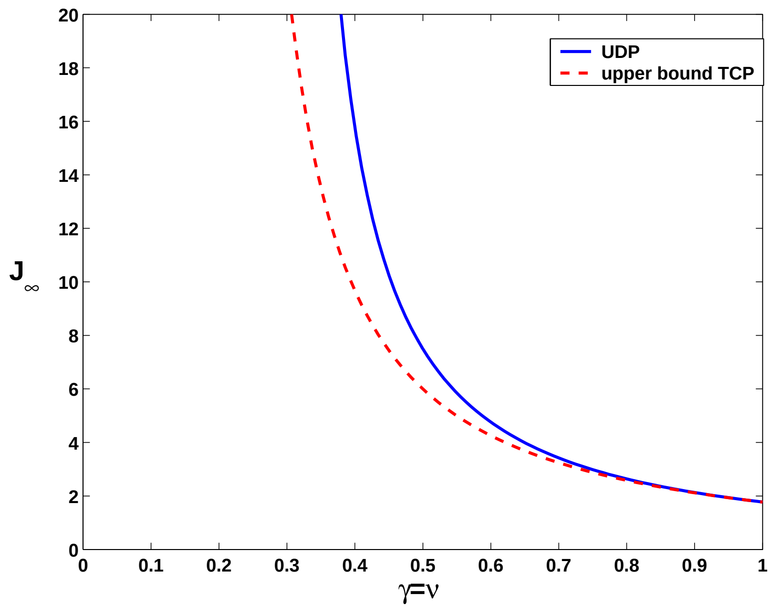

Figure 4.4: Exact infinite horizon cost using optimal LQG control under UDP-like and upper bound under TCP-like communication protocols in the scalar case.

In the scenario considered in this section when $R = 0$ and C is invertible, it is possible to directly compare the performance of optimal control under TCP-like and UDP-like protocols in terms of the infinite horizon cost J_∞^*. Let us consider for example the scalar system with the following parameters $A = 1.1, B = C = Q = W = U = 1, R = 0$. For simplicity also consider symmetric communication channels for sensor reading and control inputs, i.e. $\bar{\nu} = \bar{\gamma}$. Using results from Theorem 4.4.3 and Theorem 4.5.3 we can compute the infinite horizon cost using optimal controllers under UDP-like and an upper bound on the cost under TCP-like communication protocols, which are shown in Fig. 4.4. As expected optimal control performance under TCP-like is better than UDP-like, however the two curves are comparable for moderate packet loss. Although the TCP-like curve is only an upper bound of the true expected cost, it has been observed to be rather close to the empirical cost [68]. The observation that TCP-like and UDP-like optimal control performances seem remarkably close is extremely valuable since UDP-like protocols are much simpler to implement than TCP-like.

4.6 Conclusions

In this paper we have analyzed the LQG control problem in the case where both observation and control packets may be lost during transmission over a communication channel. This situation arises frequently in distributed systems where sensors, controllers and actuators reside in different physical locations and have to rely on data networks to exchange information. We have presented analyses of the LQG control problem under two classes of protocols: TCP and UDP. In TCP protocols, acknowledgements of successful transmissions of control packets are provided to the controller, while in UDP protocols, no such feedback is provided.

For TCP-like protocols we have solved a general LQG control problem in both the finite and infinite horizon cases. We have shown that the optimal control is a linear function of the state and that the separation principle holds. As a consequence, controller and estimator design problems are decoupled for these TCP protocols. However, unlike standard LQG control with no packet loss, the gain of the optimal observer does not converge to a steady state value. Rather, the optimal observer gain is a time-varying stochastic function of the packet arrival process. Several infinite horizon LQG controller design methodologies proposed in the literature impose time-invariance on the controller, and are therefore sub-optimal. In analyzing the infinite horizon problem, we have shown that the infinite horizon cost is bounded if and only if arrival probabilities $\bar{\gamma}, \bar{\nu}$ exceed a certain threshold. Thus, the underlying communication channel must be sufficiently reliable in order for LQG optimal controllers to stabilize the plant.

UDP-like protocols present a much more complex problem. We have shown that the lack of acknowledgement of control packets results in the failure of the separation principle. Estimation and control are now intimately coupled. We have shown that the LQG optimal control is, in general, nonlinear in the estimated state. As a consequence, the optimal control law cannot be determined explicitly in closed form, rendering this solution impractical. In the special case where the state is completed observed (C is invertible and there is no output noise i.e., $R = 0$), the optimal control is indeed linear. This special case can be viewed as one where it becomes possible to deduce whether or not the control packet was successfully transmitted. We have exhibited that the LQG optimal solution in this special case. We have shown that the the set of arrival probabilities $\bar{\gamma}, \bar{\nu}$ for which the infinite horizon cost function is bounded is smaller than the equivalent set for TCP-like protocols. However, for moderate packet loss probabilities the performance of these two classes of protocols is comparable. This makes the simpler UDP-like protocols attractrive for networked control systems.

To fully exploit UDP-like protocols it is necessary to have a controller/estimator design methodology for the general case when there is measurement noise and under partial state observation. Although the true LQG optimal controller for UDP-like protocols is time-varying and hard to compute, we might choose to determine the optimal time-invariant LQG controller. Although this is a suboptimal strategy, we believe that this controller can be determined explicitly rendering implementation simple and computationally effective. We are exploring this possibility.

This paper clearly shows that different communication protocols can affect the overall systems performance and that controller design needs to be substantially reconsidered. For example the separation principle of LQG optimal control, a milestone in classical control theory on which most modern controller design tools rely on, does not hold in general in networked control systems and in particular in control application using large scale sensor networks. Another interesting outcome of this work is that optimal controller design and communication protocol design are tightly coupled. This means that also communication protocols need rethinking at least when they are intended for real-time application such as networked control systems. Therefore, the solutions of these problems will be of paramount importance for the design of future networked control systems.

4.7 Appendix

Lemma 4.7.1. *Let $S,T \in \mathbb{M} = \{M \in \mathbb{R}^{n\times n} | M \geq 0\}$. Consider the operators $\Phi^S(S,T)$, and $\Phi^T(S,T)$ as defined in Equations (4.46) and (4.47), and consider the sequences $S_{k+1} = \Phi^S(S_k,T_k)$ and $T_{k+1} = \Phi^T(S_k,T_k)$. Consider $L^*_{S,T} = -\big(U + B'\big((1-\bar{\alpha})S + \bar{\alpha}T\big)B\big)^{-1}B'SA$. operators*

Then the following facts are true:

(a)

$$\Upsilon(S,T,L) = (1 - \frac{\bar{\nu}}{1-\bar{\alpha}})A'SA + W + \frac{\bar{\nu}}{1-\bar{\alpha}}\big(A + (1-\bar{\alpha})BL\big)'S\big(A + (1-\bar{\alpha})BL\big) + \bar{\nu}L'UL + \bar{\nu}\bar{\alpha}L'B'TBL$$

(b) $\Phi^S(S,T) = \min_L \Upsilon(S,T,L)$

(c) $0 \leq \Upsilon(S,T,L^*_{S,T}) = \Phi^S(S,T) \leq \Upsilon(S,T,L)\ \forall L$

(d) If $S_{k+1} > S_k$ and $T_{k+1} > T_k$, then $S_{k+2} > S_{k+1}$ and $T_{k+2} > T_{k+1}$.

(e) If the pair $(A, W^{1/2})$ is observable and $S = \Phi^S(S,T)$ and $T = \Phi^T(S,T)$, then $S > 0$ and $T > 0$.

Proof. Fact (a) can be easily checked by direct substitution.

(b) If U is invertible then it is easy to verify by substitution that

$$\begin{aligned}\Upsilon(S,T,L) &= \Phi^S(S,T)+\bar{\nu}(L-L^*_{S,T})'\Big(U+B'\big((1-\bar{\alpha})S+\bar{\alpha}T\big)B\Big)(L-L^*_{S,T})\\ &\geq \Phi^S(S,T)\end{aligned}$$

(c) The nonnegativeness follows form the observation that $\Upsilon(S,T,L)$ a sum of positive semi-definite matrices. In fact $(1-\frac{\bar{\nu}}{1-\bar{\alpha}})=\frac{\bar{\gamma}(1-\bar{\nu})}{\bar{\nu}+\bar{\gamma}(1-\bar{\nu})}\geq 0$ and $0\leq\bar{\alpha}\leq 1$. The equality $\Upsilon(S,T,L^*_{S,T})=\Phi^S(S,T)$ can be verified by direct substitution. The last inequality follows directly from Fact (b).

(d)

$$\begin{aligned}S_{k+2} = {}& \Phi^S(S_{k+1},T_{k+1})=\Upsilon(S_{k+1},T_{k+1},L^*_{S_{k+1},T_{k+1}})\\ \geq {}& \Upsilon(S_k,T_k,L^*_{S_{k+1},T_{k+1}})\geq\Upsilon(S_k,T_k,L^*_{S_k,T_k})\\ = {}& \Phi^S(S_k,T_k)=S_{k+1}\\ T_{k+2} = {}& \Phi^T(S_{k+1},T_{k+1})\geq\Phi^T(S_k,T_k)=T_{k+1}\end{aligned}$$

(e) First observe that $S=\Phi^S(S,T)\geq 0$ and $T=\Phi^T(S,T)\geq 0$. Thus, to prove that $S,T>0$, we only need to establish that S,T are nonsingular. Suppose they are singular, the there exist vectors $0\neq v_s\in\mathcal{N}(S)$ and $0\neq v_t\in\mathcal{N}(T)$, i.e. $Sv_s=0$ and $Tv_t=0$, where $\mathcal{N}(\cdot)$ indicates the null space. Then

$$\begin{aligned}0 = {}& v_s'Sv_s=v_s'\Phi^S(S,T)v_s=v_s'\Upsilon(S,T,L^*_{S,T})v_s\\ = {}& (1-\tfrac{\bar{\nu}}{1-\bar{\alpha}})v_s'A'SAv_s+v_s'Wv_s+\star\end{aligned}$$

where $\star$ indicates other terms. Since all the terms are positive semi-definite matrices, this implies that all the term must be zero:

$$\begin{aligned}&v_s'A'SAv_s=0\Longrightarrow SAv_s=0\Longrightarrow Av_s\in\mathcal{N}(S)\\ &v_s'Wv_s=0\Longrightarrow W^{1/2}v_s=0\end{aligned}$$

As a result, the null space $\mathcal{N}(S)$ is A-invariant. Therefore, $\mathcal{N}(S)$ contains an eigenvector of A, i.e. there exists $u\neq 0$ such that $Su=0$ and $Au=\sigma u$. As before, we conclude that Wu=0. This implies (using the PBH test) that the pair $(A,W^{1/2})$ is not observable, contradicting the hypothesis. Thus, $\mathcal{N}(S)$ is empty, proving that $S>0$. The same argument can be used to prove that also $T>0$. □

4.7.1 Proof of Theorem 4.5.2

(a)⇒(b) The main idea of the proof consists in proving convergence of several monotonic sequences. Consider the sequences $V_{k+1}=\Upsilon(V_k,Z_k,\tilde{L})$ and $Z_{k+1}=\Phi^T(V_k,Z_k)$ with initial conditions $V_0=Z_0=0$. It is easy to verify by substitution that $V_1=W+\bar{\nu}\tilde{L}'U\tilde{L}\geq 0=V_0$ and $Z_1=W\geq 0=Z_0$. Lemma 4.7.1(a) shows that the operator $\Upsilon(V,Z,\tilde{L})$ is linear and monotonically

increasing in V and Z, i.e.
$(V_{k+1} \geq V_k, Z_{k+1} \geq Z_k) \Rightarrow (V_{k+2} \geq V_{k+1}, Z_{k+2} \geq Z_{k+1})$. Also the operator $\Phi^T(V,Z)$ is linear and monotonically increasing in V and Z. Since $V_1 \geq V_0$ and $Z_1 \geq Z_0$, using an induction argument we have that $V_{k+1} \geq V_k, Z_{k+1} \geq Z_k$ for all time k, i.e. the sequences are monotonically increasing. These sequences are also bounded, in fact $(V_0 \leq \tilde{S}), (Z_0 \leq \tilde{T}) \Rightarrow (V_1 = \Upsilon(0,0,\tilde{L}) \leq \Upsilon(\tilde{S},\tilde{T},\tilde{L}) = \tilde{S}), (Z_1 = \Phi^T(0,0) \leq \Phi^T(\tilde{S},\tilde{T}) = \tilde{T})$ and the same argument can be inductively used to show that $V_k \leq \tilde{S}$ and $Z_k \leq \tilde{T}$ for all K. Consider now the sequences S_k, T_k as defined in the theorem initialized with $S_0 = T_0 = 0$. By direct substitution we find that $S_1 = W \geq 0 = S_0$ and $T_1 = W \geq 0 = T_0$. By Lemma 4.7.1(d) follows that the sequences S_k, T_k are monotonically increasing. Moreover, by Lemma 4.7.1(c) it follows that $(S_k \leq V_k, T_k \leq Z_k) \Rightarrow (S_{k+1} = \Phi^S(S_k,T_k) \leq \Upsilon(S_k,T_k,\tilde{L}) \leq \Upsilon(V_k,Z_k,\tilde{L}) = V_{k+1}), T_{k+1} = \Phi^T(S_k,T_k) \leq \Phi^T(V_k,Z_k) = Z_{k+1})$. Since this is verified for $k=0$, it inductively follows that $(S_k \leq V_k, T_k \leq Z_k)$ for all k. Finally since V_k, Z_k are bounded, we have that $(S_k \leq \tilde{S}, T_k \leq \tilde{T}$. Since $S_k, T_k)$ are monotonically increasing and bounded, it follows that $\lim_{k\to\infty} S_k = S_\infty$ and $\lim_{k\to\infty} T_k = T_\infty$, where S_∞, T_∞ are semi-definite matrices. From this it easily follows that these matrices have the property $S_\infty = \Phi^S(S_\infty, T_\infty), T_\infty = \Phi^T(S_\infty, T_\infty)$. Definite positiveness of S_∞ follows from Lemma 4.7.1(e) using the hypothesis that $(A, W^{1/2})$ is observable. The same argument can be used to prove that $T_\infty > 0$. Finally proof of uniqueness of solution and convergence for all initial conditions S_0, T_0 can be obtained similarly to Theorem 1 in [69] and it is therefore omitted.

(b)⇒(a) This part follows easily by choosing $\tilde{L} = L^*_{S_\infty,T_\infty}$, where L^* is defined in Lemma 4.7.1. Using Lemma 4.7.1(c) we have $S_\infty = \Phi^S(S_\infty,T_\infty) = \Upsilon(S_\infty,T_\infty,\tilde{L})$, therefore the statement is verified using $\tilde{S} = S_\infty$ and $\tilde{T} = T_\infty$.

4.7.2 Proof of Lemma 4.5.1

To prove the necessity condition it is sufficient to show that there exist some initial conditions $S_0, T_0 \geq 0$ for which the sequences $S_{k+1} = \Phi^S(S_k,T_k), T_{k+1} = \Phi^T(S_k,T_k)$ are unbounded, i.e. $\lim_{k\to\infty} S_k = \lim_{k\to\infty} T_k = \infty$. To do so, suppose that at some time-step k we have $S_k \geq s_k vv'$ and $T_k \geq t_k vv'$, where $s_k, t_k > 0$, and v is the eigenvector corresponding to the largest eigenvalue of A', i.e. $A'v = \lambda_{max} v$ and $|\lambda_{max}| = |A'| = |A|$. Then we have:

$$
\begin{aligned}
S_{k+1} &= \Phi^S(S_k, T_k) \geq \Phi^S(s_k vv', t_k vv') \\
&= \min_L \Upsilon(s_k vv', t_k vv', L) \\
&= \min_L \Big(s_k A'vv'A + W + 2s_k\bar{\nu}A'vv'BL + \\
&\qquad +\bar{\nu}L'\big(U + B'\big((1-\bar{\alpha})s_k vv' + \bar{\alpha}t_k vv'\big)B\big)L\Big) \\
&\geq \min_L \Big(s_k|A|^2 vv' + 2s_k\bar{\nu}\lambda_{max} vv'BL + \\
&\qquad +\bar{\nu}L'B'\big((1-\bar{\alpha})s_k vv' + \bar{\alpha}t_k vv'\big)BL\Big) \\
&= \min_L \Big(s_k|A|^2 vv' - \frac{|A|^2\bar{\nu}s_k^2}{\xi_k} vv' + \\
&\qquad +\bar{\nu}\xi_k(\lambda_{max}s_k^2 I + \frac{1}{\xi_k}BL)'vv'(\lambda_{max}s_k^2 I + \frac{1}{\xi_k}BL)\Big) \\
&\geq s_k|A|^2 vv' - \frac{|A|^2\bar{\nu}s_k^2}{(1-\bar{\alpha})s_k + \bar{\alpha}t_k} vv' \\
&= |A|^2 s_k\Big(1 - \frac{\bar{\nu}s_k}{(1-\bar{\alpha})s_k + \bar{\alpha}t_k}\Big)vv' \\
&= s_{k+1} vv'
\end{aligned}
$$

where I is the identity matrix and $\xi_k = (1-\bar{\alpha})s_k + \bar{\alpha}t_k$. Similarly we have:

$$
\begin{aligned}
T_{k+1} &= \Phi^T(S_k, T_k) \geq \Phi^T(s_k vv', t_k vv') \\
&= (1-\bar{\gamma})t_k A'vv'A + \bar{\gamma}s_k A'vv'A + W \\
&\geq (1-\bar{\gamma})t_k|A^2|vv' + \bar{\gamma}s_k|A|^2 vv' \\
&= |A|^2\big((1-\bar{\gamma})t_k + \bar{\gamma}s_k\big)vv' \\
&= t_{k+1} vv'
\end{aligned}
$$

We can summarize the previous results as follows:

$$
\begin{aligned}
&(S_k \geq s_k vv', T_k \geq t_k vv') \Rightarrow (S_{k+1} \geq s_{k+1} vv', T_{k+1} \geq t_{k+1} vv') \\
&s_{k+1} = \phi^s(s_k, t_k) = |A|^2 s_k\Big(1 - \frac{\bar{\nu}s_k}{(1-\bar{\alpha})s_k + \bar{\alpha}t_k}\Big), \\
&t_{k+1} = \phi^t(s_k, t_k) = |A|^2\big((1-\bar{\gamma})t_k + \bar{\gamma}s_k\big)
\end{aligned}
$$

Let us define the following sequences:

$$\begin{aligned} S_{k+1} &= \Phi^S(S_k,T_k), \quad T_{k+1} = \Phi^T(S_k,T_k), \quad S_0 = T_0 = vv' \\ s_{k+1} &= \phi^s(s_k,t_k), \quad t_{k+1} = \phi^t(s_k,t_k), \quad s_0 = t_0 = 1 \\ \tilde{S}_k &= s_k vv', \qquad \tilde{T}_k = t_k vv' \end{aligned}$$

From the previous derivations we have that $S_k \geq \tilde{S}_k, T_k \geq \tilde{T}_k$ for all time k. Therefore, it is sufficient to find when the scalar sequences s_k,t_k diverges to find the necessary conditions. It should be evident that also the operators $\phi^s(s,t), \phi^t(s,t)$ are monotonic in their arguments. Also it should be evident that the only fixed points of $s = \phi^s(s,t), t = \phi^t(s,t)$ are $s = t = 0$. Therefore we should be find when the origin is an unstable equilibrium point, since in this case $\lim_{k\to\infty} s_k, t_k = \infty$. Note that $t = \phi^t(s,t)$ can be written as:

$$\begin{aligned} t &= \Phi^T(s,t) = (1-\overline{\gamma})|A|^2 t + \overline{\gamma}|A|^2 s \\ &= \psi(s) = \frac{\overline{\gamma}|A|^2 s}{1-(1-\overline{\gamma})|A|^2} \end{aligned}$$

with the additional assumption $1-(1-\overline{\gamma})A^2 > 0$. A necessary condition for the stability of the origin is that the origin of restricted map $z_{k+1} = \phi(z_k, \psi(z_k))$ is stable. The restricted map is given by:

$$\begin{aligned} z_{k+1} &= |A|^2 z_k \left(1 - \overline{\nu}\frac{z_k}{(1-\overline{\alpha})z_k + \overline{\alpha}\frac{\overline{\gamma}|A|^2}{1-(1-\overline{\gamma})A^2} z_k}\right) \\ &= |A|^2 \left(1 - \frac{\overline{\nu}}{(1-\overline{\alpha}) + \overline{\alpha}\frac{\overline{\gamma}|A|^2}{1-(1-\overline{\gamma})A^2}}\right) z_k \\ &= |A|^2 \left(1 - \frac{\overline{\nu}(1-(1-\overline{\gamma})|A|^2)}{\overline{\gamma}+\overline{\nu}-\overline{\gamma\nu}-\overline{\nu}(1-\overline{\gamma})|A|^2}\right) z_k \\ &= \left(\frac{\overline{\gamma}(1-\overline{\nu})|A|^2}{\overline{\gamma}+\overline{\nu}-\overline{\gamma\nu}-\overline{\nu}(1-\overline{\gamma})|A|^2}\right) z_k \end{aligned}$$

This is a linear map and it is stable only if the term inside the parenthesis is smaller than unity, i.e.

$$\begin{aligned} \left(\frac{\overline{\gamma}(1-\overline{\nu})|A|^2}{\overline{\gamma}+\overline{\nu}-\overline{\gamma\nu}-\overline{\nu}(1-\overline{\gamma})|A|^2}\right) &< 1 \\ \overline{\gamma}(1-\overline{\nu})|A|^2 &< \overline{\gamma}+\overline{\nu}-\overline{\gamma\nu}-\overline{\nu}(1-\overline{\gamma})|A|^2 \\ |A|^2(\overline{\gamma}+\overline{\nu}-2\overline{\gamma\nu}) &< \overline{\gamma}+\overline{\nu}-\overline{\gamma\nu} \end{aligned}$$

which concludes the lemma.

4.7.3 Proof of Lemma 4.5.2

The proof is constructive. In fact we find a control feedback gain $\tilde{L}$ that satisfies the conditions stated in Theorem 4.5.2(a). Let $\tilde{L} = -\eta B^{-1}A$ where $\eta > 0$ is a positive scalar that is to be determined. Also consider $S = sI, T = tI$, where I is the identity matrix and $s, t > 0$ are positive scalars. Then we have

$$\begin{aligned}
\Upsilon(sI, tI, \tilde{L}) &= A'sA + W - 2\bar{\nu}\eta A'sA + \bar{\nu}A'B^{-'}UB^{-1}A + \\
&\quad +\bar{\nu}\eta^2 A'\big((1-\bar{\alpha})s + \bar{\alpha}t\big)A \\
&\leq |A|^2\Big(s - 2\bar{\nu}s\eta + \bar{\nu}\big((1-\bar{\alpha})s + \bar{\alpha}t\big)\eta^2\Big)I + wI \\
&= \varphi^s(s, t, \eta)I \qquad (4.56) \\
\Phi^T(sI, tI) &= \bar{\gamma}A'sA + (1-\bar{\gamma})A'tA + W \\
&\leq \big(\bar{\gamma}|A|^2 s + (1-\bar{\gamma})|A|^2 t\big)I + wI \\
&\leq \varphi^t(s, t)I \qquad (4.57)
\end{aligned}$$

where $w = |W + \bar{\nu}A'B^{-'}UB^{-1}A| > 0$ and I is the identity matrix. Let us consider the following scalar operators and sequences:

$$\begin{aligned}
\varphi^s(s, t, \eta) &= |A|^2(1 - 2\bar{\nu}\eta + \bar{\nu}(1-\bar{\alpha})\eta^2)s + \bar{\nu}\bar{\alpha}\eta^2 t + w \\
\varphi^t(s, t) &= \bar{\gamma}|A|^2 s + (1-\bar{\gamma})|A|^2 t + w \\
s_{k+1} &= \varphi^s(s_k, t_k, \eta), \;\; t_{k+1} = \varphi^t(s_k, t_k), \;\; s_0 = t_0 = 0
\end{aligned}$$

The operators are clearly monotonically increasing in s, t, and since $s_1 = \varphi^s(s_0, t_0, \eta) = w \geq s_0$ and $t_1 = \varphi^t(s_0, t_0) = w \geq t_0$, it follows that the sequences s_k, t_k are monotonically increasing. If these sequences are bounded, then they must converge to $\tilde{s}, \tilde{t}$. Therefore s_k, t_k are bounded if and only if there exist $\tilde{s}, \tilde{t} > 0$ such that $\tilde{s} = \varphi^s(\tilde{s}, \tilde{t}, \eta)$ and $\tilde{t} = \varphi^t(\tilde{s}, \tilde{t})$. Let us find the fixed points:

$$\begin{aligned}
\tilde{t} &= \varphi^t(\tilde{s}, \tilde{t}) \Rightarrow \\
\tilde{t} &= \frac{\bar{\gamma}|A|^2}{1 - (1-\bar{\gamma})|A|^2}\tilde{s} + w_t
\end{aligned}$$

where $w_t \triangleq \frac{w}{1-(1-\overline{\gamma})|A|^2} > 0$, and we must have $1-(1-\overline{\gamma})|A|^2 > 0$ to guarantee that $\tilde{t} > 0$. Substituting back into the operator φ^s we have:

$$\begin{aligned}
\tilde{s} &= |A|^2(1-2\overline{\nu}\eta+\overline{\nu}(1-\overline{\alpha})\eta^2)\tilde{s}+\overline{\nu}\overline{\alpha}\eta^2\frac{\overline{\gamma}|A|^2}{1-(1-\overline{\gamma})|A|^2}\tilde{s}+ \\
&\quad +\overline{\nu}\overline{\alpha}\eta^2 w_t + w \\
&= |A|^2\left(1-2\overline{\nu}\eta+\overline{\nu}\Big((1-\overline{\alpha})+\frac{\overline{\gamma}\overline{\alpha}|A|^2}{1-(1-\overline{\gamma})|A|^2}\Big)\eta^2\right)\tilde{s}+w(\eta) \\
&= |A|^2\left(1-2\overline{\nu}\eta+\overline{\nu}\frac{\overline{\gamma}+\overline{\nu}-\overline{\gamma\nu}-\overline{\nu}(1-\overline{\gamma})|A|^2}{1-(1-\overline{\gamma})|A|^2}\eta^2\right)\tilde{s}+w(\eta) \\
&= a(\eta)\tilde{s}+w(\eta)
\end{aligned}$$

where $w(\eta) \triangleq \overline{\nu}\overline{\alpha}\eta^2 w_t + w > 0$. For a positive solution $\tilde{s}$ to exist, we must have $a(\eta) < 1$. Since $a(\eta)$ is a convex function of the free parameter η, we can try to increase the basin of existence of solutions by choosing $\eta^* = \text{argmin}_\eta a(\eta)$, which can be found by solving $\frac{da}{d\eta}(\eta^*) = 0$ and is given by:

$$\eta^* = \frac{1-(1-\overline{\gamma})|A|^2}{\overline{\gamma}+\overline{\nu}-\overline{\gamma\nu}-\overline{\nu}(1-\overline{\gamma})|A|^2}$$

Therefore a sufficient condition for existence of solutions are given by:

$$\begin{aligned}
a(\eta^*) &< 1 \\
|A|^2\left(1-\overline{\nu}\frac{1-(1-\overline{\gamma})|A|^2}{\overline{\gamma}+\overline{\nu}-\overline{\gamma\nu}-\overline{\nu}(1-\overline{\gamma})|A|^2}\right) &< 1 \\
\left(\frac{\overline{\gamma}(1-\overline{\nu})|A|^2}{\overline{\gamma}+\overline{\nu}-\overline{\gamma\nu}-\overline{\nu}(1-\overline{\gamma})|A|^2}\right) &< 1
\end{aligned}$$

which is the same bound for the necessary condition of convergence in Lemma 4.5.1.

If this condition is satisfied then $\lim_{k\to\infty} s_k = \tilde{s}$ and $\lim_{k\to\infty} t_k = \tilde{t}$. Let us consider now the sequences $\overline{S_k} = s_k I$, $\overline{T_k} = t_k I$, $S_{k+1} = \Upsilon(S_k, T_k, \tilde{L})$ and $T_{k+1} = \Phi^T(S_k, T_k)$, where $\tilde{L} = -\eta^* B^{-1}A$, $S_0 = T_0 = 0$, and s_k, t_k where defined above. These sequences are all monotonically increasing. From Equations (4.56) and (4.57) it follows that $(S_k \leq s_k I, T_k \leq t_k I) \Rightarrow (S_{k+1} =\leq s_{k+1} I, T_{k+1} \leq t_k I)$. Since this is verified for $k = 0$ we can claim that $S_k < \tilde{s}I$ and $T_k < \tilde{t}I$ for all k. Since S_k, T_k are monotonically increasing and bounded, then they must converge to positive semidefinite matrices $\tilde{S}, \tilde{T} \geq 0$ which solve the equations $\tilde{S} = \Upsilon(\tilde{S}, \tilde{T}, \tilde{L})$ and $\tilde{T} = \Phi^T(\tilde{S}, \tilde{T})$. Since by hypothesis the pair $(A, W^{1/2})$ is observable, using similar arguments of Lemma 4.7.1(e), it is possible to show that $\tilde{S}, \tilde{T} > 0$. Therefore $\tilde{S}, \tilde{T}, \tilde{L}$ satisfy the conditions of statement (a) Theorem 4.5.2, from which if follows statement (b) of the same theorem. This implies that the sufficient conditions derived here guarantee the claim of the lemma.