CDS 140a Winter 2014 Homework 3

From MurrayWiki

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ACM 101b/AM 125b/CDS 140a, Winter 2014

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Turn in to box outside Steele House

(PDF)

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. **Perko, Section 2.2, problem 5** (only if you have had AM 125a or CDS 201): Let V be a normed linear space. If $T : V \to V$ satisfies
$$\|T(u) - T(v)\| \leq c\|u - v\|$$
for all $u, v \in V$ with $0 < c < 1$ then T is called a *contraction mapping*. It can be shown that contraction mappings give rise to unique solutions of the equation $T(u) = v$:

   Theorem (Contraction Mapping Principle) Let V be a complete normed linear space and $T : V \to V$ a contraction mapping. Then there exists a unique $u \in V$ such that $T(u) = v$.

   Let $f \in C^1(E)$ and $x_0 \in E$. For $I = [-a, a]$ and $u \in C(I)$, let
$$T(u)(t) = x_0 + \int_0^t f(u(s))ds.$$
Define a closed subset V of $C(I)$ and apply the Contraction Mapping Principle to show that the integration equation (7) in Perko, Section 2.2 has a unique solution $u(t)$ for all $t \in [-a, a]$ provided the constant $a > 0$ is sufficiently small.

2. **Perko, Section 2.2, problem 3** (if you have not had ACM 125a or CDS 201): Use the method of successive approximations to show that if $f(x, t)$ is continuous in t in some interval containing $t = 0$ and continuously differentiable in x for all x in some open set $E \subset \mathbb{R}^n$ containing x_0, then there exists $a > 0$ such that the initial value problem
$$\dot{x} = f(x, t), \qquad x(0) = x_0$$
has a unique solution $x(t)$ on the interval $[-a, a]$.
   - Note: this problem is very similar to the case we worked out in class (and that you can find in Perko), but now $f(x, t)$ depends on the time t. If you get stuck, there is a hint in Perko (not transcribed here).
3. **Perko, Section 2.3, problem 1**: Use the fundamental theorem for linear systems in Chapter 1 of Perko to solve the initial value problem
$$\dot{x} = Ax, \qquad x(0) = y.$$
Let $u(t, y)$ denote the solution and compute

$$\Phi(t) = \frac{\partial u}{\partial y}(t, y).$$

Show that $\Phi(t)$ is the fundamental matrix solution of

$$\dot{\Phi} = A\Phi, \qquad \Phi(0) = I.$$

- Note: this problem works through the more general result for nonlinear systems (Corallary on page 83) for the special case of a linear system.

4. **Perko, Section 2.5, problem 4**: Sketch the flow of the linear system

$$\dot{x} = Ax \quad \text{with} \quad A = \begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix}$$

and describe $\phi_t(N_\epsilon(x_0))$ for $x_0 = (-3, 0), \epsilon = 0.2$.

5. **Perko, Section 2.5, problem 5**: Determine the flow $\phi_t : \mathbb{R}^2 \to \mathbb{R}^2$ for the nonlinear system

$$\dot{x} = f(x) \quad \text{with} \quad f(x) = \begin{bmatrix} -x_1 \\ 2x_2 + x_1^2 \end{bmatrix}$$

and show that the set $S = \{x \in \mathbb{R}^2 | x_2 = -x_1^2/4\}$is invariant with respect to the flow ϕ_t.

6. **Perko, Section 2.6, problem 2**: Classify the equilibrium points of the Lorenz equation

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ \mu x_1 - x_2 - x_1 x_3 \\ x_1 x_2 - x_3 \end{bmatrix}$$

for $\mu > 0$. At what value of the parameter μ do two new equilibrium points "bifurcate" from the equilibrium point at the origin?

- Note: the number and/or stability type of equilibrium points will change depending on the value of μ. Make sure to classify the equilibrium points for different ranges of μ and not just one value of μ. If you get stuck, there are some hints in problem 1(e) of Perko.

Notes:

- The problems are transcribed above in case you don't have access to Perko. However, in the case of discrepancy, you should use Perko as the definitive source of the problem statement.