Packet-based Control: the UDP-like case

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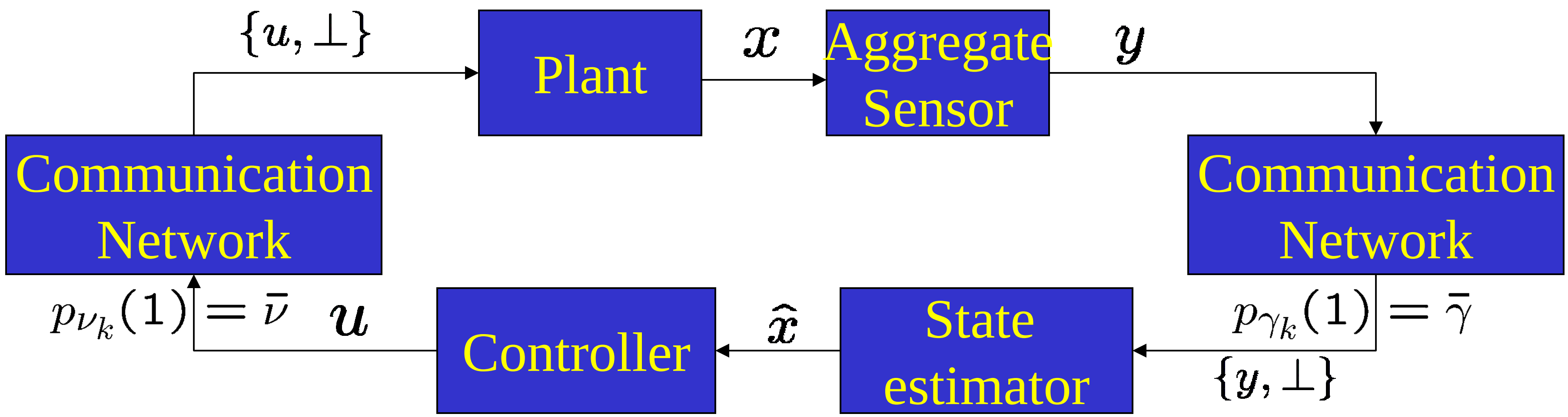
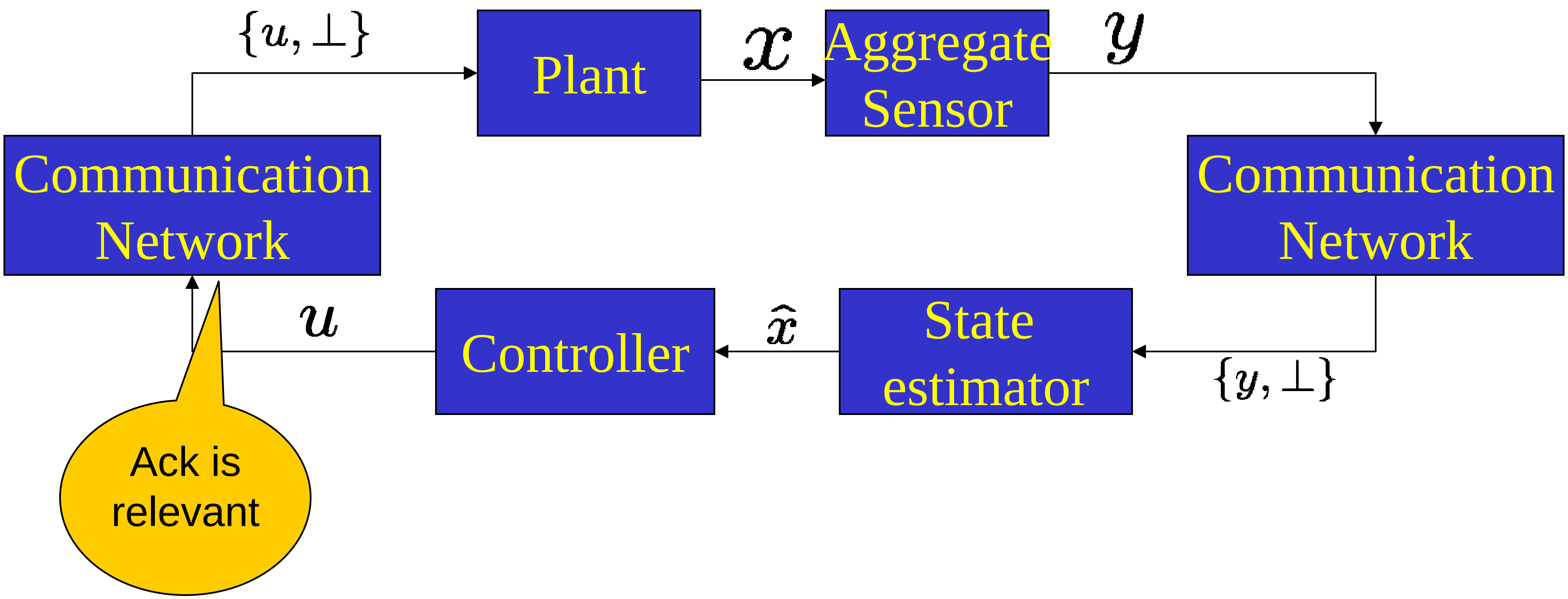
The control problem



- What is the minimum arrival probability that guarantees "acceptable" performance of estimator and controller?
- How is the arrival rate related to the system dynamics?
- Can we design estimator and controller independently?
- Are the optimal estimator and controller still linear?
- Can we provide design guidelines?

LQG control with intermittent observations and control



We'll group all communication protocols in two classes:
TCP-like (acknowledgement is available)
UDP-like (acknowledgement is absent)

Summary of results: TCP case

- Finite horizon LQG:
  - The separation principle holds for TCP-like protocols
  - The optimal estimator is the time-varying Kalman Filter
  - The optimal control is linear
  - The optimal cost can be bounded
- Infinite Horizon LQG:
  - There exist critical values γ_c, ν_c below which the closed loop system fails to stabilize the system, and above which we can bound the state
  - We can compute this bound

UDP case: Estimator Design

Prediction Step

$$
\begin{aligned}
\hat{x}_{k+1|k} &= A\hat{x}_{k|k} + \bar{\nu}Bu_k \\
P_{k+1|k} &= AP_{k|k}A' + Q + \bar{\nu}(1-\bar{\nu})Bu_k u_k' B'
\end{aligned}
$$

Correction Step

$$
\begin{aligned}
\hat{x}_{k+1|k+1} &= \hat{x}_{k+1|t} + \gamma_{k+1}K_{k+1}(y_{k+1} - C\hat{x}_{k+1|t}) \\
P_{k+1|k+1} &= P_{k+1|t} - \gamma_{k+1}P_{k+1|t}C'(CP_{k+1|t}C' + R)^{-1}CP_{k+1|t} \\
K_{k+1} &= P_{k+1|t}C'(CP_{k+1|t}C' + R)^{-1}
\end{aligned}
$$

LQG Controller Design: UDP-like case

Scalar system, i.e. x∈R

$$
\begin{aligned}
x_{t+1} &= x_t + \nu_t u_t \\
y_t &= x_t + v_t, \qquad \text{cov}(v_t) = \gamma_t 1 + (1-\gamma_t)\sigma^2 I
\end{aligned}
$$

$$J_N = \mathbb{E}[\sum_{t=0}^{N} |x_t|^2]$$

$$\mathcal{I}_t = \{y_t, \gamma_t, \ldots\}$$

t=N

$$V_N(x_N) = \mathbb{E}[x_N^2]$$

t=N-1

$$
\begin{aligned}
V_{N-1}(x_{N-1}) &= \min_{u_{N-1}} \mathbb{E}[x_{N-1}^2 + V_N(x_N) \mid \mathcal{I}_{N-1}] \\
&= \min_{u_{N-1}} \mathbb{E}[x_{N-1}^2 + x_N^2 \mid \mathcal{I}_{N-1}] \\
&= \min_{u_{N-1}} \left(\mathbb{E}[x_{N-1}^2 + x_{N-1}^2 \mid \mathcal{I}_{N-1}] + \bar{\nu} u_{N-1}^2 + 2\bar{\nu} u_{N-1} \hat{x}_{N-1|N-1} \right) \\
&= 2\,\mathbb{E}[x_{N-1}^2 \mid \mathcal{I}_{N-1}] - \bar{\nu} \hat{x}_{N-1|N-1}^2, \qquad u_{N-1}^* = -\hat{x}_{N-1|N-1} \\
&= (2-\bar{\nu})\,\mathbb{E}[x_{N-1}^2 \mid \mathcal{I}_{N-1}] + \bar{\nu} P_{N-1|N-1}
\end{aligned}
$$

LQG Controller Design: UDP-like case

t=N-2

$$
\begin{aligned}
V_{N-2}(x_{N-2}) &= \min_{u_{N-2}} \mathbb{E}[x_{N-2}^2 + V_{N-1}(x_{N-1}) \mid \mathcal{I}_{N-2}] \\
&= \mathbb{E}[(3-\bar{\nu})x_{N-2}^2 \mid \mathcal{G}_{N-2}] + \bar{\gamma} + \bar{\nu}(2-\bar{\gamma})P_{N-2|N-2} + \\
&\quad + \min_{u_{N-2}} \Big((2\bar{\nu} - \bar{\nu}^3 - \bar{\nu}^2\bar{\gamma} + \bar{\nu}^3\bar{\gamma})u_{N-2}^2 + 2\bar{\nu}(2-\bar{\nu})\hat{x}_{N-2|N-2}u_{N-2} + \\
&\quad + \bar{\nu}\bar{\gamma}\frac{1}{P_{N-2|N-2} + \bar{\nu}(1-\bar{\nu})u_{N-2}^2 + 1} \Big)
\end{aligned}
$$

$$
u_t^* = k_t(y_t, \ldots, y_0, \gamma_t, \ldots, \gamma_0, P_0, x_0)
$$

NONLINEAR FUNCTION OF INFORMATION SET *It*

General Case:

- The separation principle does not hold for UDP-like protocols
- The optimal control feedback is in general a nonlinear function of the information set

Finite horizon LQG UDP-like controller: Special case: C invertible, R=0

- The value function can be written as:

$$V_k(x_k) = \mathbb{E}[x_k' S_k x_k | \mathcal{G}_k] + \text{trace}\Big((T_k - S_k) P_{k|k}\Big) + \text{trace}(D_k Q)$$

- The optimal control feedback is linear function of the state estimate

$$\begin{aligned} u_k^* &= -(U_k + B'((1-\bar{\alpha}) S_{k+1} + \bar{\alpha} T_{k+1}) B)^{-1} B' S_{k+1} A \, \widehat{x}_{k|k} \\ &= L_k \, \widehat{x}_{k|k} \end{aligned}$$

- The optimal minimal cost is:

$$J_N^* = \overline{x}_0' S_0 \overline{x}_0 + \text{trace}(S_0 P_0) + \sum_{k=1}^{N} \text{trace}(((1 - \bar{\gamma}) T_k + \bar{\gamma} S_k) Q)$$

UDP-like Infinite Horizon

- Necessary condition for the convergence of

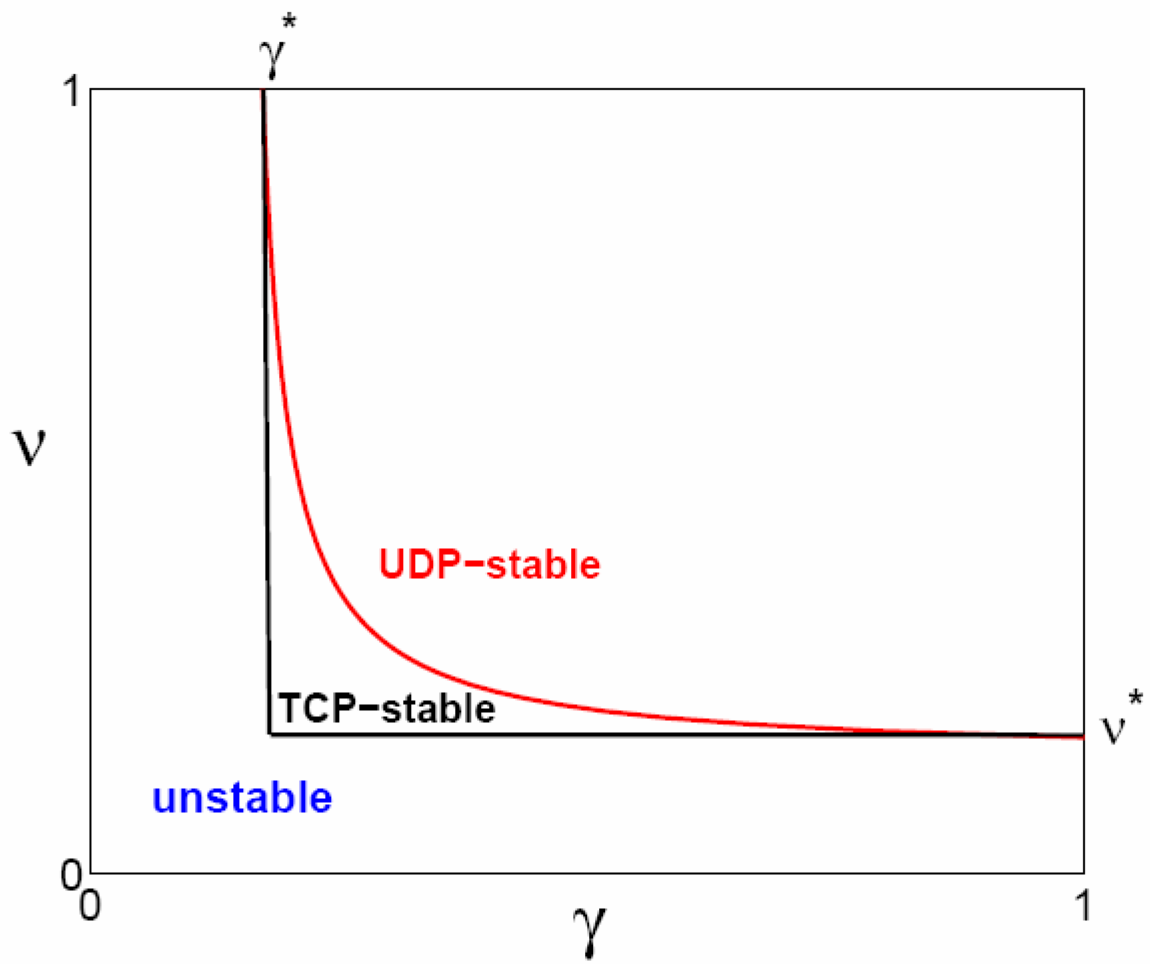
$$\lim_{k \to \infty} S_k = S_\infty, \quad \lim_{k \to \infty} T_k = T_\infty$$

  is that

$$|A|^2(\bar{\gamma} + \bar{\nu} - 2\bar{\gamma}\bar{\nu}) < \bar{\gamma} + \bar{\nu} - \bar{\gamma}\bar{\nu}$$

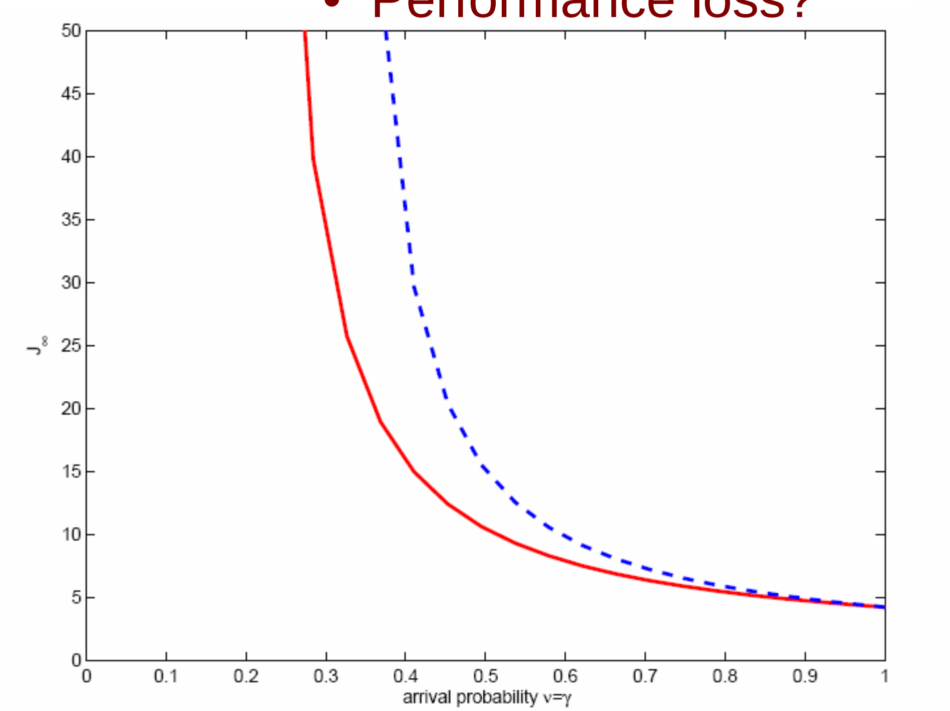
- If B is invertible the condition is sufficient

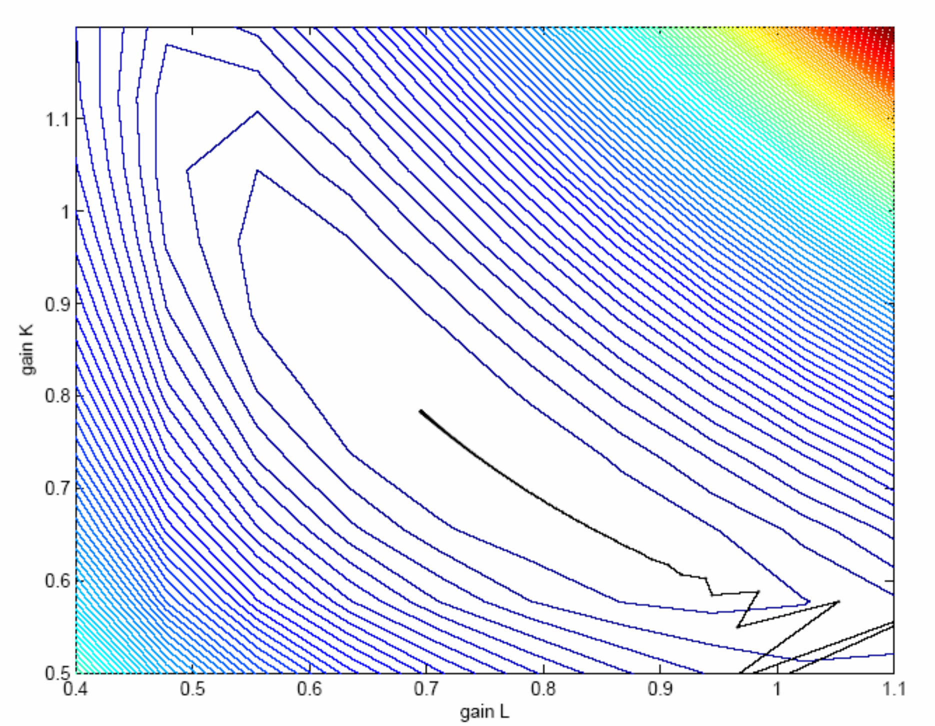
Infinite horizon LQG UDP-like controller: Special case: C invertible, R=0



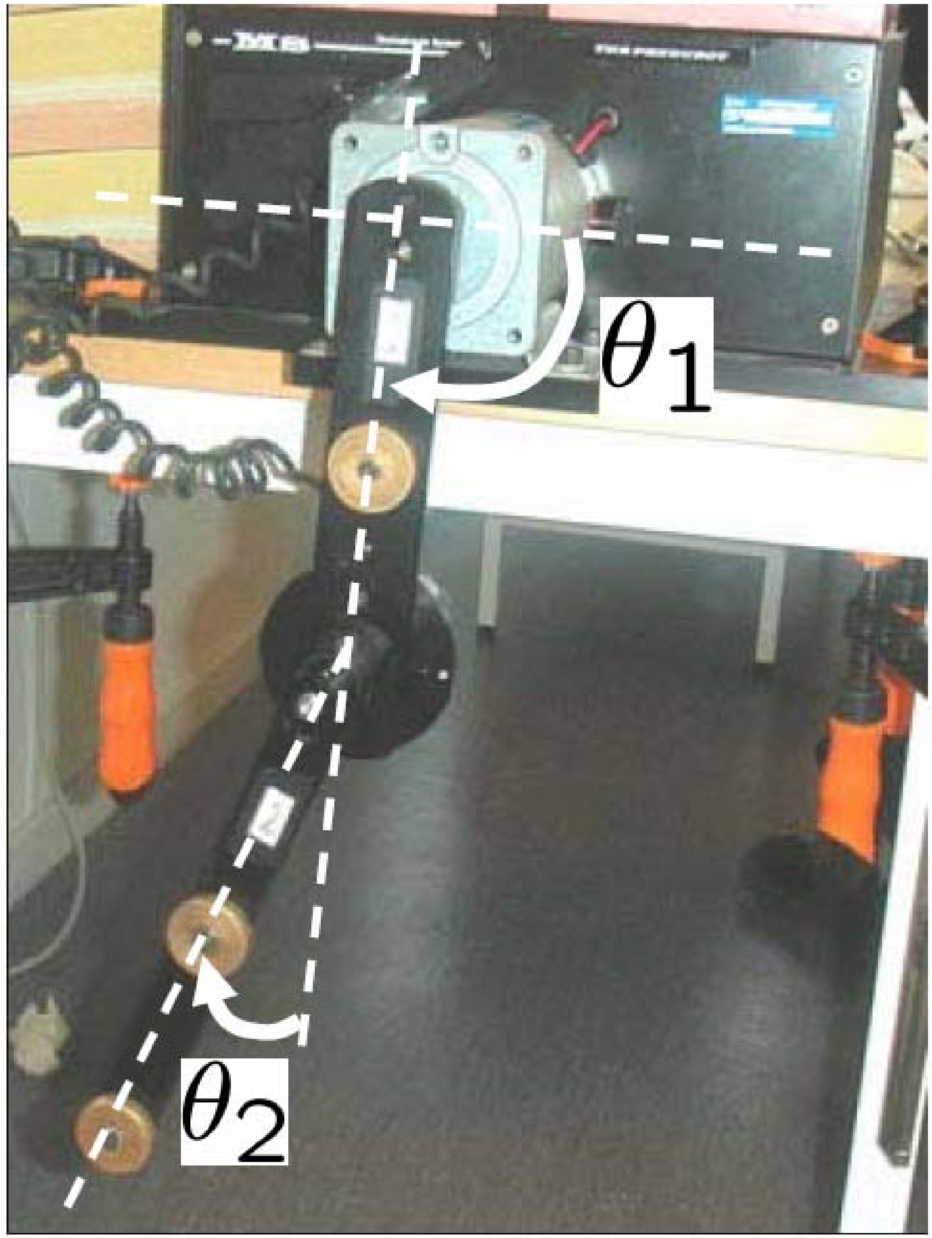
Optimal LQG restricted to linear static controllers

- Concentrate on UDP design
  - What is the optimal linear time-invariant controller
    - Great practical value
    - Optimization?
    - Cost convergence
    - Performance loss?





Example: Pendubot



Courtesy of Mechatronics Inc.

The model: linearized and sampled (T=0.005s)

$$A = \begin{bmatrix} 1.001 & 0.005 & 0.000 & 0.000 \\ 0.35 & 1.001 & -0.135 & 0.000 \\ -0.001 & 0.000 & 1.001 & 0.005 \\ -0.375 & -0.001 & 0.590 & 1.001 \end{bmatrix}, B = \begin{bmatrix} 0.001 \\ 0.540 \\ -0.002 \\ -1.066 \end{bmatrix}$$

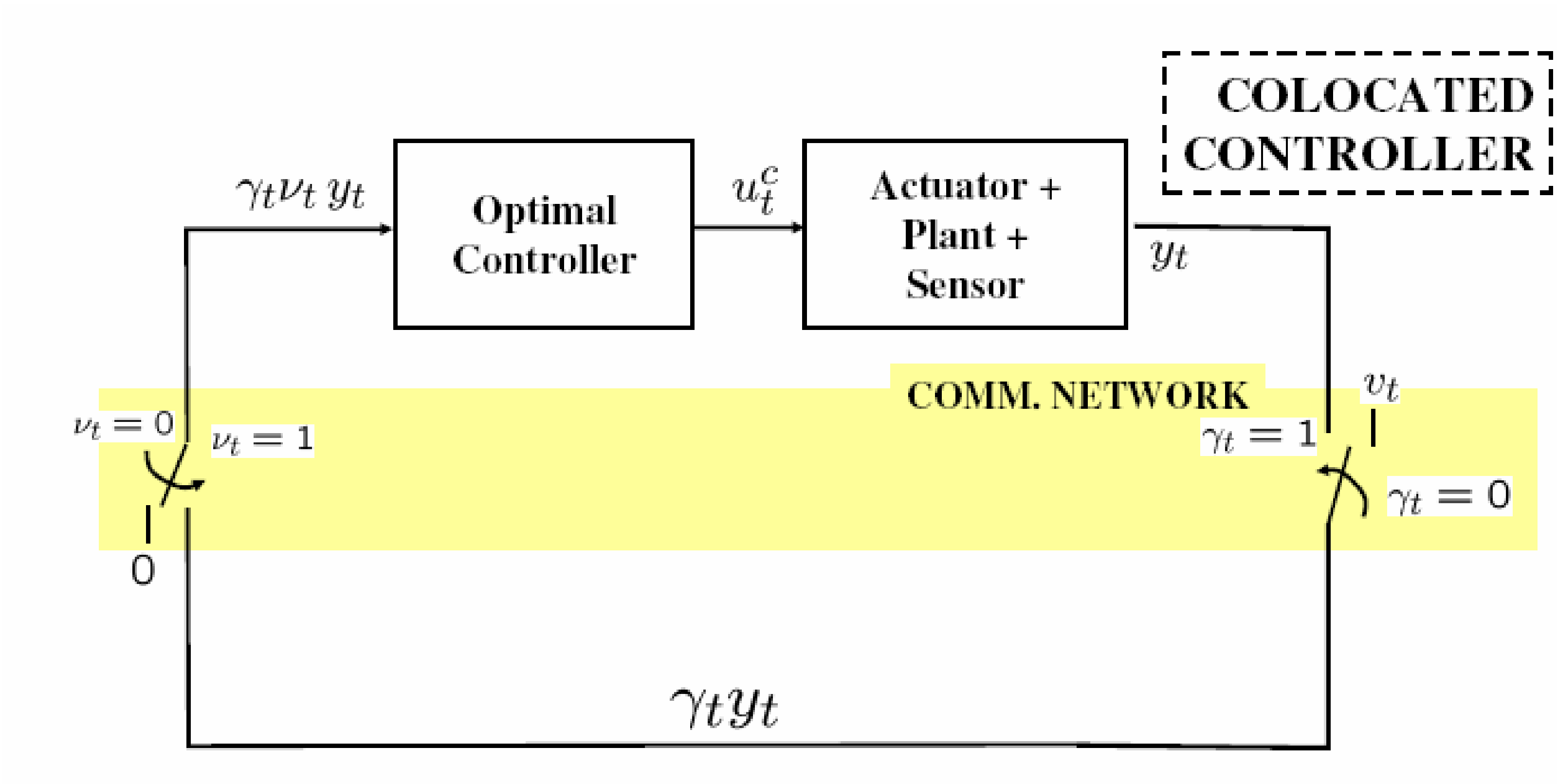
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \ R = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.001 \end{bmatrix}, \ U = 2$$

$$Q = qq^T, q = \begin{bmatrix} 0.003 \\ 1.000 \\ -0.005 \\ -2.150 \end{bmatrix}, W = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

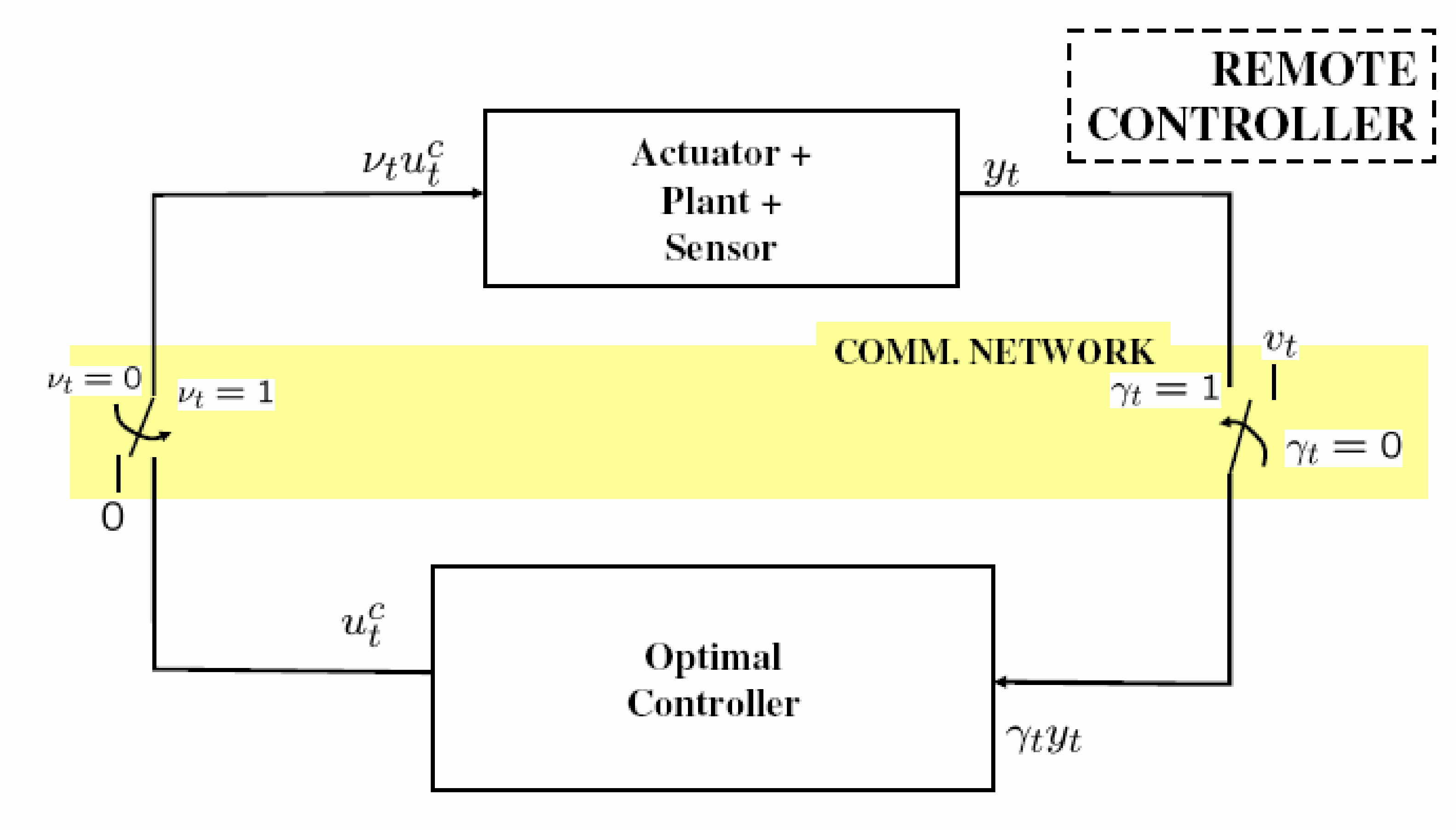
$$\text{eig}(A) = (1.061, 1.033, 0.968, 0.941)$$

the pairs (A, B) and (A, Q) are controllable

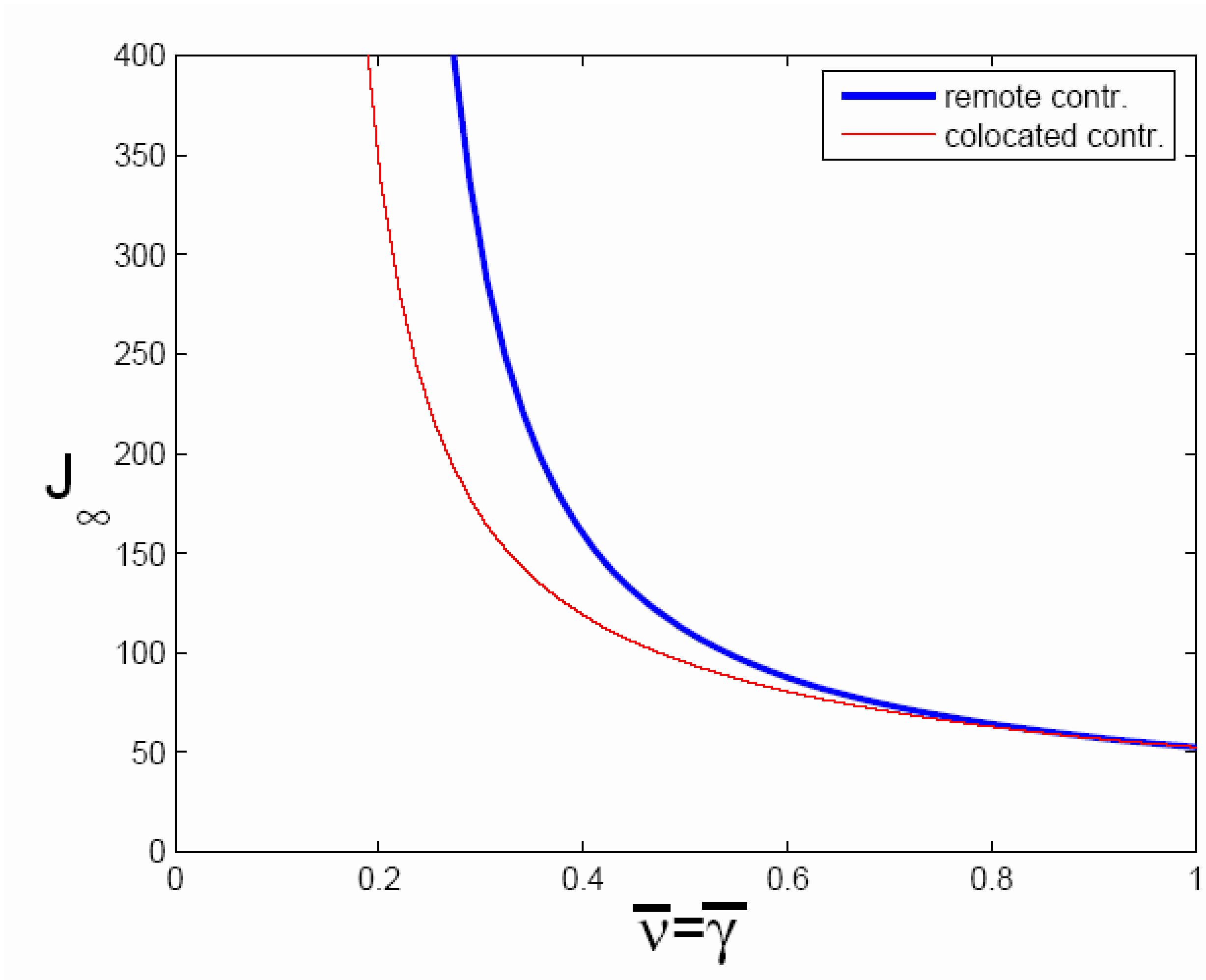
Co-located controller



Remote controller

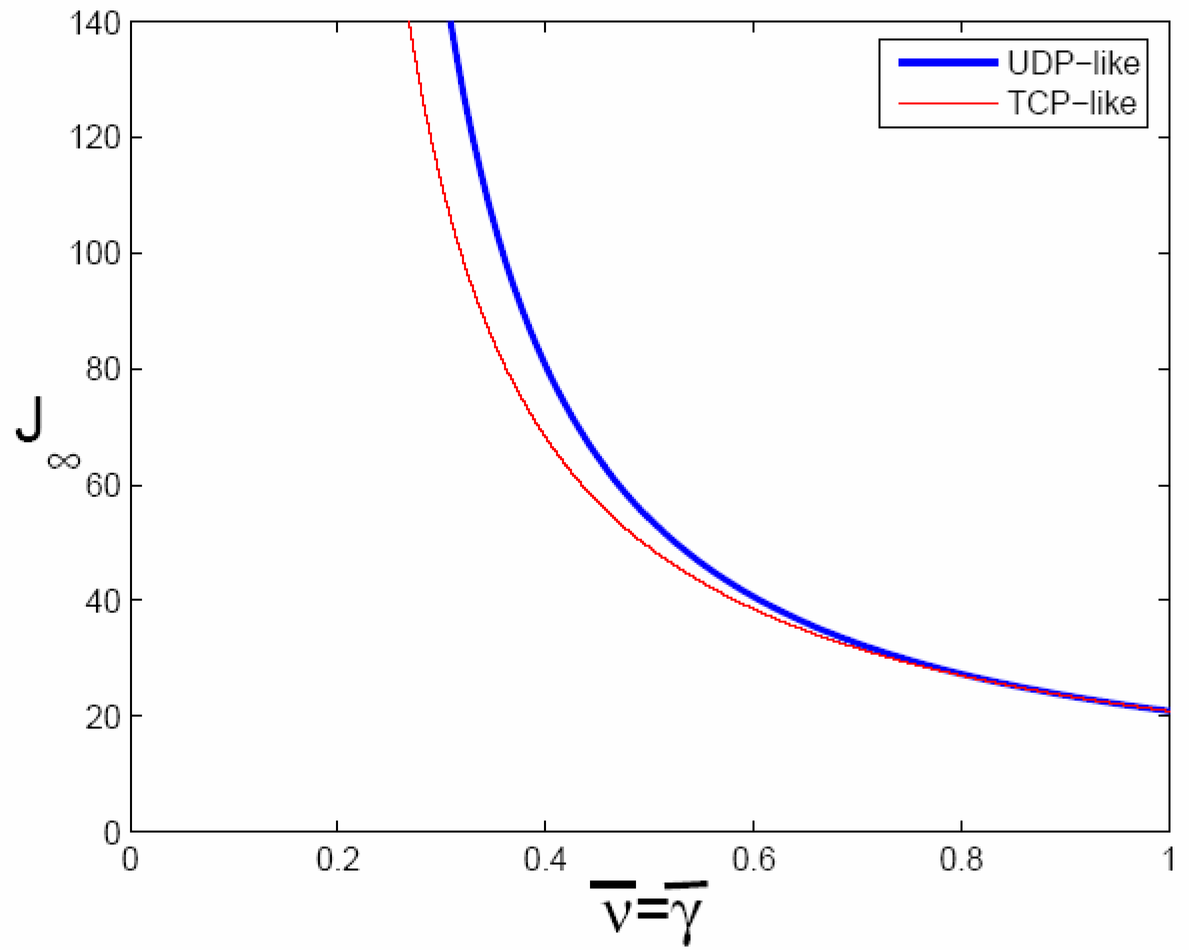


Comparison

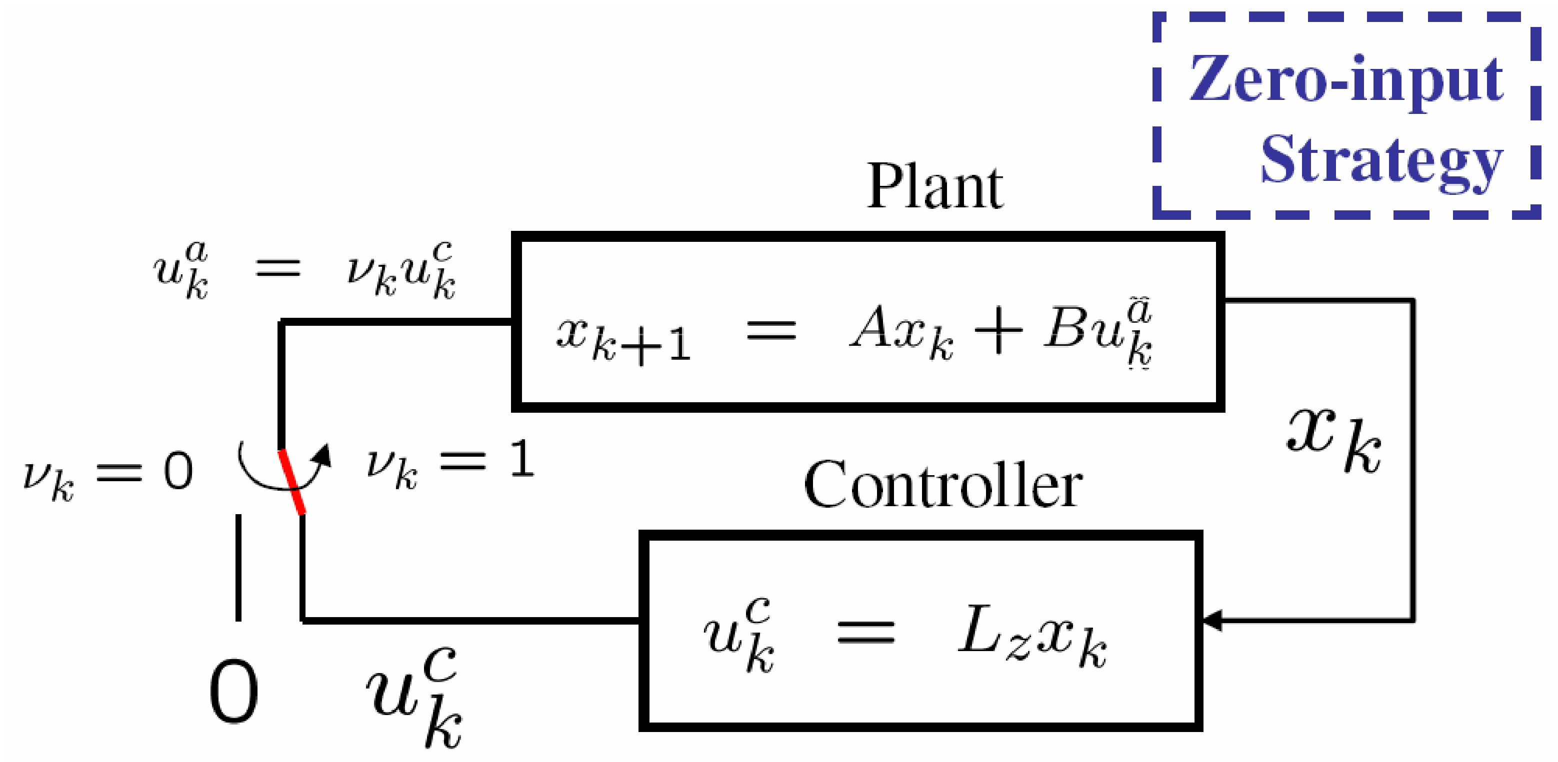


400
350
300
250
200
150
100
50
0
0
0.2
0.4
0.6
0.8
1
J∞
ν̄=γ̄
remote contr.
colocated contr.

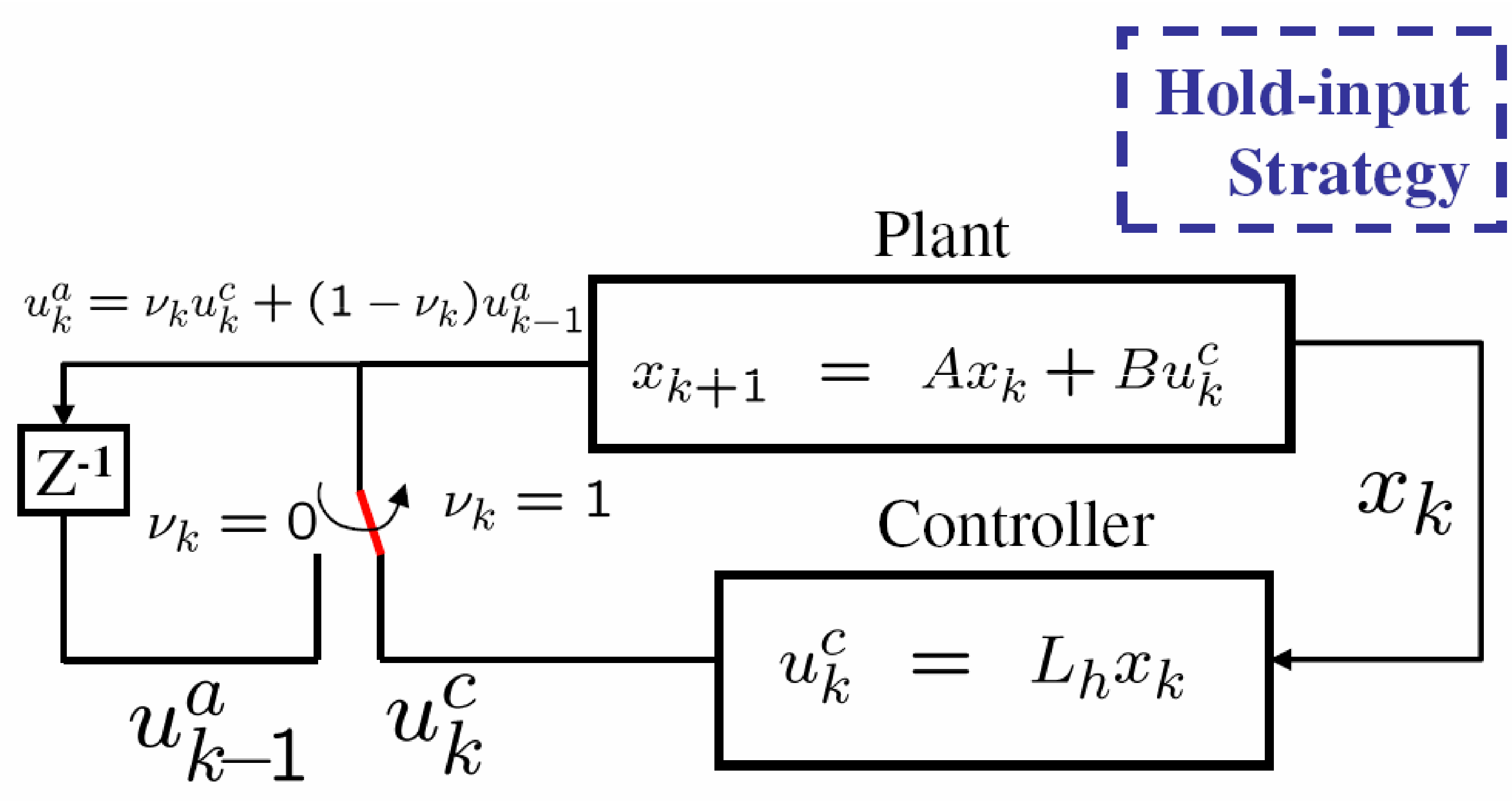
TCP-like vs UDP-like (R=0, C Invertible)



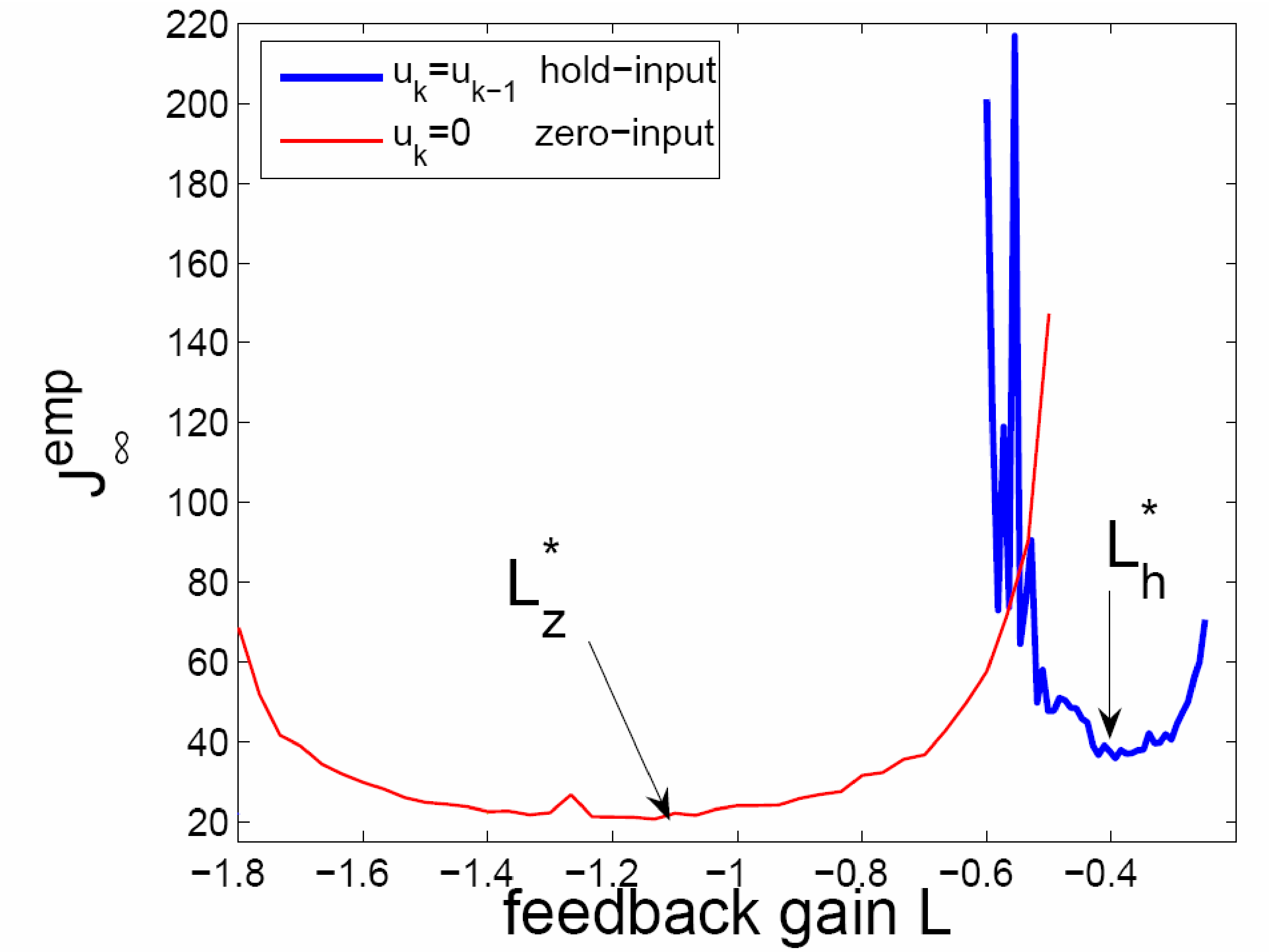
Second example: scalar noiseless system



Second example: scalar noiseless system



Comparison



u_k=u_{k-1} hold−input
u_k=0 zero−input
220
200
180
160
140
120
100
80
60
40
20
J_∞^{emp}
L_z^*
L_h^*
−1.8
−1.6
−1.4
−1.2
−1
−0.8
−0.6
−0.4
feedback gain L

Fundamental tradeoffs

- **TCP – Transmission Control Protocol**
  - PRO: feedback information on packet delivery
  - CONS: more expensive to implement
  - Conjecture: Easier control design
- **UDP – User Datagram Protocol**
  - PRO: simpler communication infrastructure
  - CONS: less information available
  - Conjecture: complex control design

Conclusions

- Control over networks presents degrees of novelty
- Characterized LQG problem for two classes of protocols:
  - TCP-like
    - Separation principle applies
    - FH and IH solved
    - Optimal control is linear and can be computed in closed form
    - Transition to instability studied for IH appears
    - Critical values for this transition are given
  - UDP-like
    - Harder problem
    - No separation principle
    - Optimal controller is nonlinear function of the information state
    - Special case provides a linear controller

For the designer

- Analysis is a great tool
- We now have a clearer picture of the fundamental tradeoffs
- Choose where to place your complexity
  - Communication design (ack-based vs. nack)
  - Control Design (linear vs. nonlinear)
  - Sensor design (Low noise, state observation)