Linear Algebra and ODEs review

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1 Linear Algebra

1.1 Eigenvalues and eigenvectors

Consider the square matrix $A \in \mathbb{R}^{n \times n}$. (v, λ) are an (eigenvector, eigenvalue) pair of matrix A if and only if $Av = \lambda v$, $v \in R^{n \times 1}$, $v \neq 0$, $\lambda \in \mathbb{C}$.

Remark 1. *Real matrices may have complex eigenvalues. They occur in conjugate pairs.*

We find eigenvalues by setting $\det(A - \lambda \mathbb{I}_{n \times n}) = 0$, where $\mathbb{I}_{n \times n}$ is the identity matrix, and solving for λ. Once we know the eigenvalue, let's call it λ_i, we solve for the corresponding eigenvector(s) from the equation $(A - \lambda_i)v_i = 0$.

Remark 2. *Upper- or lower- triangular and diagonal matrices have their eigenvalues on the diagonal. Matrix*

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

has eigenvalues 1 and 3 because it's upper triangular.

MATLAB is excellent at working with matrices. The command $[V, D] = eig(A)$ returns diagonal matrix D with eigenvalues on its diagonal and matrix V whose columns are the corresponding eigenvectors, so that A V = V D.

1.2 Rank and nullity

Definition 1. *The vectors $v_1, \ldots, v_m \in \mathbb{R}^{n \times 1}$ are called linearly dependent if there exist scalars (numbers) $a_1, \ldots, a_m \in \mathbb{R}$, not all equal to 0, such that $a_1 \cdot v_1 + \ldots + a_m \cdot v_m = 0$. When no such scalars exist, the vectors $v_1, \ldots, v_m$ are called linearly independent.*

We can consider the columns of matrix A as a set of n vectors, as above.

Definition 2. *The column rank of matrix A is the number of linearly independent columns of the matrix. The row rank of matrix A is the number of linearly independent rows of the matrix. The column rank = the row rank. The notation for the rank is rank(A).*

For matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 4 \\ 3 & 0 & 6 \end{bmatrix}$$

the rank is 1.

Definition 3. *A vector v is in the nullspace of matrix A if $Av = 0$. The dimension of the nullspace of A is called nullity, notation nullity(A). It equals the number of linearly independent vectors in the nullspace.*

Theorem 1. *The rank-nullity theorem: rank(A) + nullity(A) = n.*

Using the rank-nullity theorem, the nullity of matrix A from the last example is 2. We find two linearly independent vectors in the nullspace of A: $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Can you give another example of two linearly independent vectors in the nullspace of A?

1.3 The matrix exponential

The exponential of matrix A is $e^A = \sum_{i=0}^{\infty} \frac{A^i}{i!} = \mathbb{I}_{n\times n} + A + \frac{1}{2}A^2 + \dots$

Properties:
1. e^A is also an $n \times n$ matrix
2. $\frac{d}{dt}e^{At} = Ae^{At}$
3. $e^A e^{-A} = \mathbb{I}_{n\times n}$
4. $e^{A+B} \neq e^A e^B$, unless $AB = BA$
5. If A is diagonal with elements $d_1, \dots, d_n$ on its diagonal, then e^A is also a diagonal matrix with elements $e^{d_1}, \dots, e^{d_n}$ on its diagonal.

In MATLAB, use function expm to compute the matrix exponential. Don't use exp, they are not the same!

2 ODEs

2.1 Chain rule and change of variables

Theorem 2. *If y is a function of u, $y = f(u)$ and u is a function of x, $u = g(x)$, then*

$$\left.\frac{dy}{dx}\right|_{x=c} = \left.\frac{dy}{du}\right|_{u=g(c)} \cdot \left.\frac{du}{dx}\right|_{x=c} \tag{1}$$

Homework 1 problem 2b (CDS 101) and 3b (CDS 110): We introduce the new variables by rescaling time. Use the chain rule to adjust the time scale.

2.2 First order linear ODEs

The simplest first order linear ODE is $x'(t) = ax(t)$, where x and a are scalars and a is a constant. The solution is $x(t) = e^{at}x(0)$.

The general form of a first order linear ODE is:

$$x'(t) + p(t)x(t) = g(t), \tag{2}$$

where p and g are known functions (that may be constant). The solution method is due to Leibniz: We multiply Equation (2) by the *integrating factor* $\mu(t) = e^{\int p(t)dt}$. Then Equation (2) becomes $\frac{d}{dt}(\mu(t)x(t)) = \mu(t)g(t)$. Thus, the general solution is:

$$x(t) = \frac{1}{\mu(t)} \int \mu(t)g(t)dt + c, \tag{3}$$

where c is a constant and $\mu(t) = e^{\int p(t)dt}$.

When x is a vector, the solution to the first order ODE $x'(t) = Ax(t)$ is $x(t) = e^{At}x(0)$.

2.3 Second order linear ODEs

2.3.1 Second order linear homogeneous ODEs

The form of a *homogeneous* (the right hand side equals 0) second order linear ODEs is given by

$$\frac{d^2z(t)}{dt^2} + a\frac{dz(t)}{dt} + bz(t) = 0, \tag{4}$$

where a and b are numbers.

The solution to this equation may be obtained by *assuming* that $z(t) = e^{st}$. Then, by substituting the solution we guessed into the equation, we obtain that $s^2e^{st} + ase^{st} + be^{st} = 0$. Since the exponential cannot be zero, we divide it out and obtain

$$s^2 + as + b = 0. \tag{5}$$

This is called the *characteristic polynomial* associated with a homogeneous second order ODE. It is a quadratic equation with the two roots (not always distinct) $s_1 = -\frac{a}{2} + \frac{\sqrt{a^2-4b}}{2}$ and $s_2 = -\frac{a}{2} - \frac{\sqrt{a^2-4b}}{2}$.

There are two possible solutions to Equation (4): e^{s_1t} and e^{s_2t}. Since the two solutions are linearly independent when the discriminant $a^2 - 4b \neq 0$ and Equation (4) is linear, then the solution is their linear combination

$$z(t) = c_1e^{s_1t} + c_2e^{s_2t}, \tag{6}$$

where c_1 and c_2 are real numbers. If we knew two specified initial condition (an example would be knowing $z(0)$ and its first derivative $z'(0)$), then we could also determine the values of c_1 and c_2.

There are 3 cases for the form of $z(t)$, according to the value of the discriminant in Equation (5):

1. When $a^2-4b > 0$, then s_1 and s_2 are both real numbers and $z(t) = e^{-\frac{at}{2}}(c_1e^{\frac{1}{2}\sqrt{a^2-4b}t} + c_2e^{-\frac{1}{2}\sqrt{a^2-4b}t})$.

2. When $a^2 - 4b = 0$, then $s_1 = s_2 = -\frac{a}{2}$. The solutions e^{s_1t} and e^{s_2t} are linearly dependent here, so we only know one of the solutions to Equation (4). Since Equation (4) is a second order ODE, it must have two solutions. We find the missing solution by *assuming* another form that the solution might have, $z(t) = A(t)e^{st}$. By substituting our guess back into Equation(4), we obtain $\frac{d^2A(t)}{dt^2} + (2s_1 + a)\frac{dA}{dt} + (s_1^2 + as_1 + b)A(t) = 0$. Since both terms $s_1 + a = 0$ and $s_1^2 + as_1 + b = 0$, it must be that $\frac{d^2A(t)}{dt^2} = 0$. Hence, $A(t) = t$ and $z(t) = c_1e^{-\frac{at}{2}} + c_2te^{-\frac{at}{2}}$.

3. When $a^2-4b < 0$, then α_1 and α_2 are complex conjugates of each other. Following algebraic manipulation that we do not include here, $z(t) = e^{-\frac{at}{2}}(B_1\sin(\frac{1}{2}\sqrt{4b-a^2})t + B_2\cos(\frac{1}{2}\sqrt{4b-a^2})t)$, where B_1 and B_2 are constants.

Let's apply our results to a damped mass-spring system. It is a second degree homogeneous equation for the position of the mass $y(t)$ as a function of time:

$$m\frac{d^2y(t)}{dt^2} + c\frac{dy(t)}{dt} + ky(t) = 0. \tag{7}$$

It doesn't have the form we gave for a second order homogeneous ODE because the first term is multiplied by the mass. If we divide by m and let $a = \frac{c}{m}$ and $b = \frac{k}{m}$, then we are in the preferred form.

The solutions to Equation (7) are split in 3 cases:

1. When $c^2 - 4mk > 0$ (overdamping), the solution is $y(t) = e^{-\frac{c}{2m}t}(c_1e^{-\Omega t} + c_2e^{\Omega t})$, where $\Omega = \sqrt{c^2-4mk}/2m$. In the overdamped case, there is no oscillation and the amplitude of $y(t)$ eventually decays to equilibrium.

2. When $c^2 - 4mk = 0$ (critical damping), the solution is $y(t) = e^{-\frac{c}{2m}t}(c_1 + c_2 t)$. In critical damping, $y(t)$ decays to equilibrium as quickly as possible with no oscillation.

3. When $c^2 - 4mk < 0$ (under-damping), the solution is $y(t) = e^{-\frac{c}{2m}t}(c_1 \cos(\Omega t) + c_2 \sin(\Omega t))$, where $\Omega = \sqrt{c^2 - 4mk}/2m$. In under-damping, $y(t)$ returns to equilibrium by oscillating with decreased amplitude.

Fore more information on the damped mass-spring system and its solutions, see [2].

2.3.2 Derivative trick

We can reduce a second order linear homogeneous ODE to a first order linear ODE by introducing a dummy variable. In fact, we can also reduce higher order linear homogeneous ODEs to a first order linear ODE using the same trick with more than one dummy variable.

In the damped mass-spring example, we introduce dummy variable $z(t) = y'(t) \Rightarrow z'(t) = y''(t)$. By replacing $y''(t)$ in Equation (7), we obtain $z'(t) = -\frac{c}{m}z(t) - \frac{k}{m}y(t)$. We now stack the two equations and obtain $\begin{pmatrix} y'(t) \\ z'(t) \end{pmatrix} = \begin{pmatrix} z(t) \\ -c/mz(t) - k/my(t) \end{pmatrix} \Rightarrow \begin{pmatrix} y'(t) \\ z'(t) \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \begin{pmatrix} y(t) \\ z(t) \end{pmatrix}$.

We now form the vector $v(t) = \begin{pmatrix} y(t) \\ z(t) \end{pmatrix}$ and we get $v'(t) = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} v(t)$. This is a first order linear ODE, so the solution is $v(t) = e^{At}v(0)$, where $A = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix}$. Thus, $\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} = e^{At} \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix}$.

2.3.3 Second order linear non-homogeneous ODEs

Non-homogeneous ODEs have the form

$$\frac{d^2 z(t)}{dt^2} + a\frac{dz(t)}{dt} + bz(t) = f(t), \tag{8}$$

where a and b are numbers, f is a *forcing term.*

The solution to Equation (8) has two components:

$$z(t) = z_h(t) + z_p(t), \tag{9}$$

where z_h is the solution of the corresponding homogeneous equation $\frac{d^2 z_h}{dt^2} + a\frac{dz_h}{dt} + bz_h(t) = 0$ and z_p is the *particular solution.* There are no fixed rules for determining the particular solution z_p. However, it is usually a good guess to assume that z_p is similar function to the forcing term f. We provide a list of common examples below. The coefficients for $f(t)$ are known, whereas the coefficients for $z_p(t)$ are unknown and must be determined.

1. $f(t)$ is a polynomial of order n: $f(t) = at^2 + bt + c$ (order 2 here), then $z_p(t)$ is a polynomial of order n: $z_p(t) = a_1 t^2 + a_2 t + a_3$ (order 2 here).

2. $f(t)$ is a trigonometric function: $f(t) = a\sin(st)$ or $f(t) = b\cos(st)$ or $f(t) = a\sin(st) + b\cos(st)$, then $z_p(t)$ is a combination of *both* trigonometric functions: $z_p(t) = a_1 \sin(st) + a_2 \cos(st)$.

3. $f(t)$ is an exponential function: $f(t) = ae^{st}$, then $z_p(t)$ is an exponential function: $z_p(t) = a_1 e^{st}$

4. $f(t)$ is a linear combination of the functions above, then $z_p(t)$ is a linear combination of the functions above with unknown coefficients.

Let's solve the damped mass-spring system with forcing function $f(t) = \sin(\omega t)$. ω is known here. Then the equation we want to solve is:

$$\frac{d^2y(t)}{dt^2} + \frac{c}{m}\frac{dy(t)}{dt} + \frac{k}{m}y(t) = \sin(\omega t). \tag{10}$$

We know that $y(t) = y_h(t) + y_p(t)$. $y_h(t)$ is the solution of the homogeneous second order ODE and we can determine it as in Section 2.3.1. We then guess the form of the particular solution $y_p(t) = s_1 \sin(\omega t) + s_2 \cos(\omega t) \Rightarrow y_p'(t) = s_1\omega \cos(\omega t) - s_2\omega \sin(\omega t), y''(t) = -s_1\omega^2 \sin(\omega t) - s_2\omega^2 \cos(\omega t)$.

By replacing them in Equation (10), we obtain that $\sin(\omega t)(-s_1\omega^2 - \frac{c}{m}s_2\omega + \frac{k}{m}s_1 - 1) + \cos(\omega t)(-s_2\omega^2 + \frac{c}{m}s_1\omega + s_2\frac{k}{m}) = 0$. Since $\sin(\omega t)$ and $\cos(\omega t)$ are linearly independent functions (try proving the linear independence), it must be that their coefficients are 0.

Hence, $-s_1\omega^2 - \frac{c}{m}s_2\omega + \frac{k}{m}s_1 - 1 = 0$ and $-s_2\omega^2 + \frac{c}{m}s_1\omega + s_2\frac{k}{m} = 0$. This is a linear equation system with two equations and two unknowns s_1 and s_2. We do not include its solution here.

References

[1] http://www.engr.sjsu.edu/trhsu/Chapter 4 Second order DEs.pdf

[2] https://en.wikipedia.org/wiki/Damping