Studying the Logistic Map and the Mandelbrot Set using SOS Methods

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Outline

1. **The Logistic Map**
   - Idea of proof: an example
   - Invariance Proofs
   - Proof Length ("Complexity")
2. **Mandelbrot Set**
   - Inner and Outer Bounds
   - Fragility in the Mandelbrot Set

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1. **The Logistic Map**
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   - Invariance Proofs
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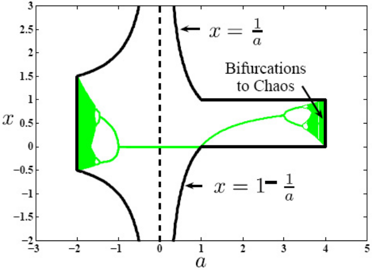
2. **Mandelbrot Set**
   - Inner and Outer Bounds
   - Fragility in the Mandelbrot Set

What is the Logistic Map?

Defining Equation

$$x_{k+1} = ax_k(1 - x_k)$$

$$\textit{where} \quad a, x_k \in \mathbb{R}$$



- Complex dynamics and bifurcation to chaos
- Allows us to visualize dynamics vs. parameter in 2D
- Fixed points at $x = 0$ and $x = 1 - \frac{1}{a}$
- Bifurcation occurs at $a = 1$ (fixed points interchange stability properties)

Region of Attraction

$$0 \le x \le 1 \qquad 1 \le a \le 4$$

$$1 - \frac{1}{a} \le x \le \frac{1}{a} \qquad 0 \le a \le 1$$

$$\frac{1}{a} \le x \le 1 - \frac{1}{a} \qquad -2 \le a \le 0.$$

Example ($-2 \leq a < 1$ Branch)

Proving invariance is equivalent to proving

$$\left\{ \begin{array}{r} a^2(2x-1)^2 \leq (a-2)^2 \\ (a-1)(a+2) \leq 0 \\ a^2(2ax(1-x)-1)^2 > (a-2)^2 \end{array} \right\} = \varnothing$$

$$\textit{Setting} \quad a = 2 \quad \left\{ \begin{array}{r} 4-(2x-1)^2 \geq 0 \\ 4-(-4x(1-x)-1)^2 < 0 \end{array} \right\}$$

we can manipulate the second equation to show that

$$4-(-4x(1-x)-1)^2 = (2x-1)^2(4-(2x-1)^2) < 0$$

A contradiction! The set is empty

Useful expressions

Definition

Given polynomials $\{f_1, \ldots, f_s\} \in \mathbb{R}[\mathbf{x}]$ the *Algebraic Cone* generated by the f_i's is the set

$$\mathbf{C}(f_1, \ldots, f_s) = \left\{ f \,\middle|\, f = \lambda_0 + \sum_i \lambda_i F_i \right\}$$

where $F_i \in \mathbf{M}(f_1, \ldots, f_s)$, λ_i's are SOS Polynomials

Definition

The *subset of the cone* is the set of F_i's in the definition of the **Cone.**

Definition

The *proof order* is the degree of the highest order term in the P-satz refutation.

Definition

The *SOS multiplier order* is the order of each of the λ_i's in the **Cone.**

Branch 1: $-2 \leq a \leq 1$

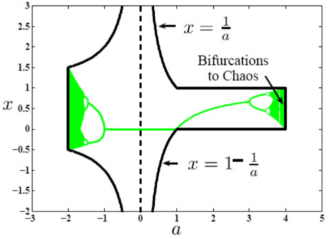
The Constraint Set

$$f_1(a,x) = (a-2)^2 - a^2(2x-1)^2 \geq 0$$

$$f_2(a,x) = -(a+2)(a-1) \geq 0$$

$$f_3(a,x) = a^2(2ax(1-x)-1)^2 - (a-2)^2 \geq 0$$

$$f_3(a,x) \neq 0$$



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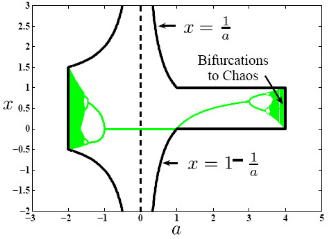

Want SOS polynomials $p_0,\ p_i,\ p_{ij},\ p_{ijk}$

$$-(f_3^\alpha)^2 = p_0 + \sum_i p_i f_i + \sum_{\{i,j\}} p_{ij} f_i f_j + \sum_{\{i,j,k\}} p_{ijk} f_i f_j f_k \qquad \alpha \in \{0,1,2\dots\}$$

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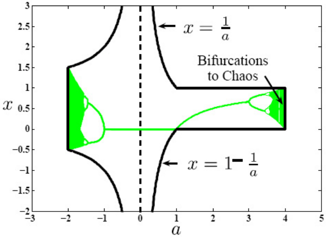


Form of the Refutation

$$-f_3^2 = p_{13} f_1 f_3 + p_{123} f_1 f_2 f_3$$

where $p_{13}(a,x) = \frac{4}{3} - \frac{2}{3}a + \frac{1}{3}a^2 - xa^2 + x^2a^2, \quad p_{123}(a,x) = \frac{1}{3}.$

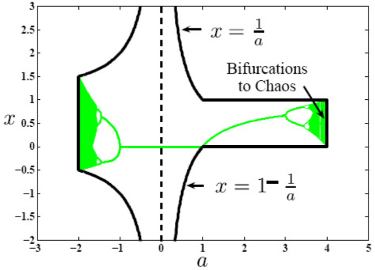
Note $\quad p_{13} = \frac{1}{3} f_2 + a(x^2a - ax + 1) \quad$ *and* $\quad f_3 = -a(ax^2 - xa + 1) f_1$

Branch 2: $1 \leq a \leq 4$

Sign problems!

The Constraint Set

$$f_1(a,x) = (2x-1)^2 - 1 \leq 0$$
$$f_2(a,x) = (a-1)(a-4) \leq 0$$
$$f_3(a,x) = 1 - (2ax(1-x)-1)^2 \leq 0$$
$$f_3(a,x) \neq 0.$$



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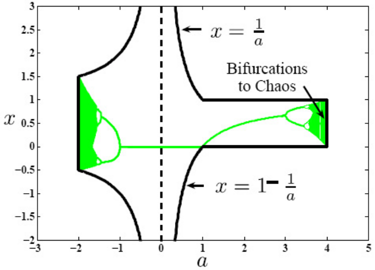

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Note $\quad p_{13} = -\frac{1}{3} f_2 + (a^2x^2 - xa^2 + 1)$ *and* $f_3 = -f_1(a^2x^2 - xa^2 + 1)$

How to define/classify 'Proof Length'

- Order of the Proof and/or Order of the SOS Multipliers
- Size and Conditioning of the SDP

Example (Order of the Proof)

For the $1 \leq x \leq 4$ an alternative refutation can is:

$$-f_3^2 = p_0 + p_1 f_1 + p_2 f_2 + p_3 f_3$$

Polynomial	Order in x	Order in a
p_0	8	4
p_1	6	4
p_2	8	2
p_3	4	2

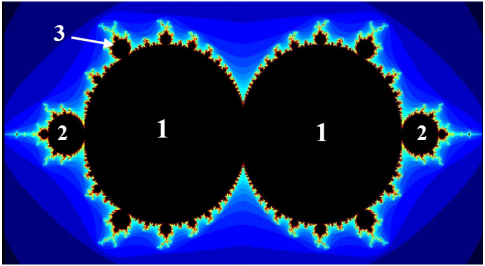
- Proofs same order but use different subsets of the cone.
- This proof is linear in f_i's but the SOS multipliers more complicated.
- Which proof is longer?
- Size and Conditioning of the SDP may be a more natural choice BUT are implementation dependent!

Outline

What is the Mandelbrot Set?

The λ Parameterization

$$z_{k+1} = \lambda z_k(1 - z_k)$$
$$\textit{where} \quad \lambda, z_k \in \mathbb{C}$$



- The complex version of the logistic map
- Fixed points at $z = 0$ and $z = 1 - \frac{1}{\lambda}$
- $\lambda \in \textit{Mset} \Leftrightarrow z_k$ *bounded*
- Color indicates no. iterations to unboundedness (interpretation "distance" from Mset)
- Important to note that Mandelbrot set is a subset of parameter space not dynamical system space

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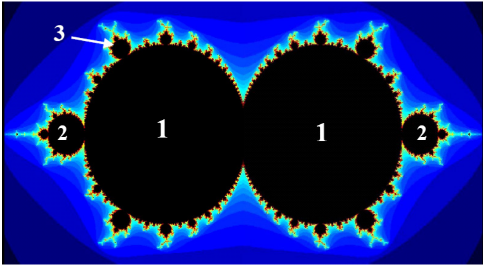
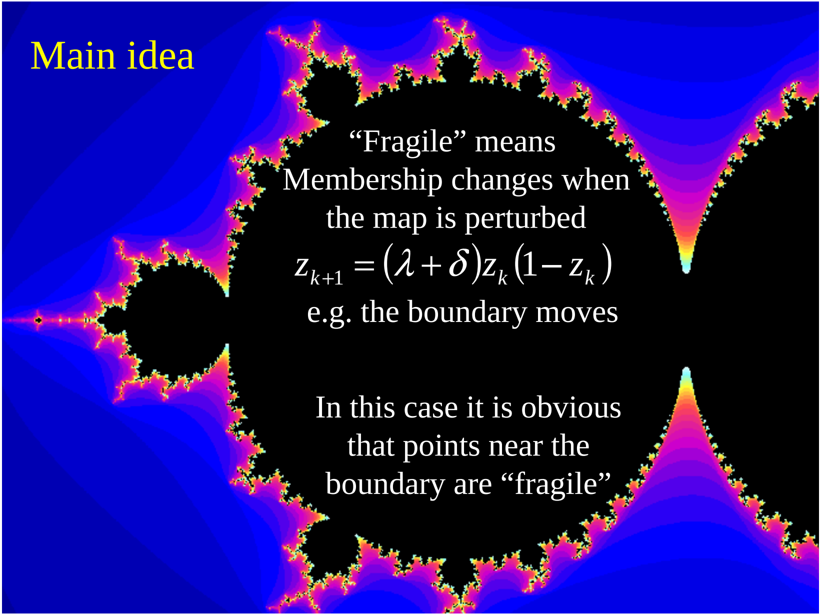


- **Set membership is undecidable in the sense of Turing**
- Classic computational problem that is easily visualized.
- Most computational problems involve uncertain dynamical systems, from protein folding to complex network analysis. Not easily visualized.
- Natural questions are typically computationally intractable, and conventional methods provide little encouragement that this can be systematically overcome.

Fragility In the Mandelbrot Set



Main idea

"Fragile" means
Membership changes when
the map is perturbed

$$z_{k+1} = (\lambda + \delta) z_k (1 - z_k)$$

e.g. the boundary moves

In this case it is obvious
that points near the
boundary are "fragile"

Cyclic Lobes: Regional ("Global") Proofs

$$z_{k+1} = \lambda z_k(1 - z_k)$$

$$V(z_k) = |z_k|^2$$

$$\textit{Stability} \quad \Leftrightarrow V(z_k) \geq V(z_{k+1})$$

$$\Leftrightarrow |z_k|^2 - |\lambda z_k(1 - z_k)|^2 \geq 0$$

$$\Leftarrow 1 \geq |\lambda||(1 - z_k)|$$

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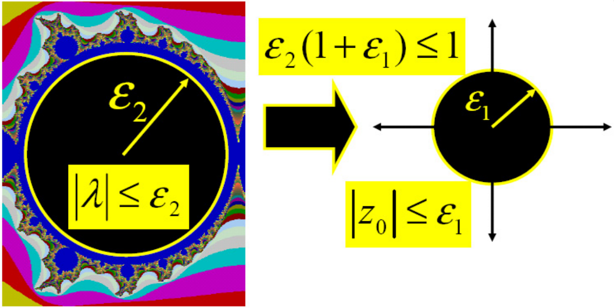

$$\{\lambda \leq \mathbf{1}\} \subset \ \mathit{Mset}$$

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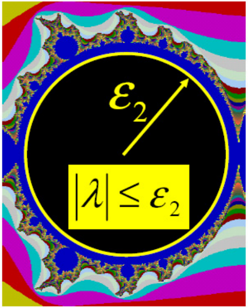
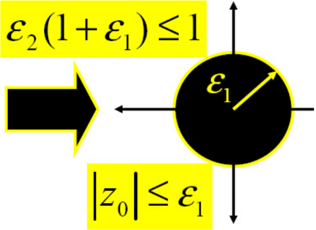
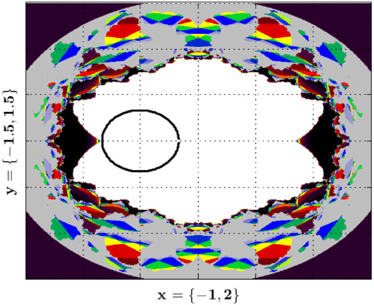




Julia Sets for $|\lambda| = 0.75$



$$\{\lambda \leq 1\} \subset \; \mathit{Mset}$$

The Left Lobe

Fixed point at $z = (1 - \frac{1}{\lambda})$

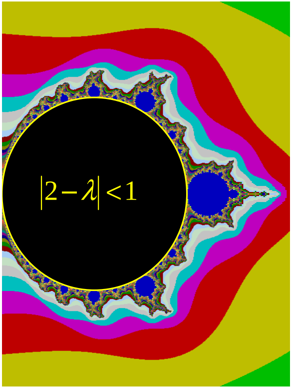
let $\quad w_k = z_k - z^* \quad$ then

$$w_{k+1} = w_k(2 - \lambda - \lambda w_k)$$

Using a similar Lyapunov Function

$$V(w_k) = |w_k|^2$$

$$|w_{k+1}|^2 \leq |w_k|^2 \Leftarrow |2-\lambda|+|\lambda||w_k| \leq 1$$



$\{|2 - \lambda| \leq 1\} \subset \ Mset$

Regional ('Global') in λ Local in z

- The 2-period map is

$$Q(z) = z_{k+2} = \lambda z_{k+1}(1 - z_{k+1})$$
$$= \lambda^2 z_k(1 - z_k)(1 - \lambda z_k + \lambda z_k^2)$$

- The fixed points of this map are

$$z_1^* = 0, z_2^* = 1 - \frac{1}{\lambda}$$
$$z_{3,4}^* = \frac{\lambda + 1 \pm \sqrt{\lambda^2 - 2\lambda - 3}}{2\lambda}$$

For an attracting fixed point

$$\left|\dot{Q}\right| < 1$$

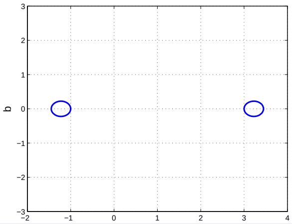
Using z_3^*

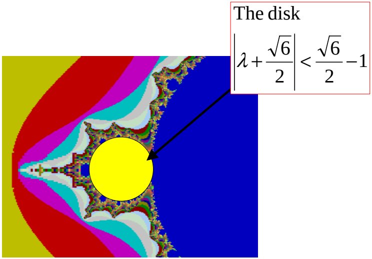
$$\dot{Q}(z_3^*) = \frac{d}{dx} F(F(z))_{z=z_3^*}$$
$$= F'(F(z_3^*))F'(z_3^*)$$
$$= F'(z_4^*)F'(z_3^*)$$
$$= 4 + 2\lambda - \lambda^2$$

Therefore the 2-cycle is locally attracting for $|4 + 2\lambda - \lambda^2| < 1$.

2 Period Lobes

Letting $\lambda = a + bi$ gives
$(4 - a^2 + 2a + b)^2 + (2b - 2ab)^2 < 1.$





Want to show that:

$$\left|\lambda + \frac{\sqrt{6}}{2}\right| < \frac{\sqrt{6}}{2} - 1 \subseteq (4 - a^2 + 2a + b)^2 + (2b - 2ab)^2 - 1 < 0$$

This is equivalent to showing that;

$$\left\{ \begin{array}{c} (4 - a^2 + 2a + b)^2 + (2b - 2ab)^2 - 1 \geq 0 \\ \left(\frac{\sqrt{6}}{2} - 1\right)^2 - \left(a + \frac{\sqrt{6}}{2}\right)^2 + b^2 > 0 \end{array} \right\} = \varnothing$$

2 Period Lobes

Constraint Set

$$f_1 = (4 - a^2 + 2a + b)^2 + (2b - 2ab)^2 - 1 \geq 0$$

$$f_2 = \left(\frac{\sqrt{6}}{2} - 1\right)^2 - \left(a + \frac{\sqrt{6}}{2}\right)^2 - b^2 - \varepsilon \geq 0$$

Positivstellensatz refutation

$$p_0 + p_1 f_1 + p_2 f_2 = -1$$

$p_1 \simeq 395$ and $p_2 = 4465.4 + 667.03a^2 - 1974.1a + 1223.3b^2$

- Determining set membership for local z values in the two period region required an increase in both the order and the size of the proof.
- The proof is also ill conditioned.
- These differences are associated with an increase in proof length or 'complexity'.

2 Period Lobes

Constraint Set

$$f_1 = (4 - a^2 + 2a + b)^2 + (2b - 2ab)^2 - 1 \geq 0$$

$$f_2 = \left(\frac{\sqrt{6}}{2} - 1\right)^2 - \left(a + \frac{\sqrt{6}}{2}\right)^2 - b^2 - \varepsilon \geq 0$$

Positivstellensatz refutation (increasing ε)

$$p_0 + p_1 f_1 + p_2 f_2 = -1$$

$p_1 = 19.51$ and $p_2 = 223.48 + 49.25a^2 - 112.24a + 68.32b^2$

- Moving further away from the boundary (less fragile region) improves conditioning.
- This is good evidence that SDP conditioning should be part of proof length definition.

2 Period Lobes

Constraint Set

$$f_1 = (4 - a^2 + 2a + b)^2 + (2b - 2ab)^2 - 1 \geq 0$$

$$f_2 = \left(\frac{\sqrt{6}}{2} - 1\right)^2 - \left(a + \frac{\sqrt{6}}{2}\right)^2 - b^2 - \varepsilon \geq 0$$

Positivstellensatz refutation 2 (setting $\varepsilon = 0$)

$$p_0 + p_1 f_1 + p_3 f_1 f_2 = -f2^2$$

$$p_2 = 1.4b^4 + a^4 + 4.8a^3 + 2.9a^2b^2 + 7ab^2 + 8.6a^2 + 6.9a + 4.1b^2 + 2$$
$$p_3 = 1.2a^2b^2 - .4ab^2 + .3a^2 + .94b^2 + .35a + .34$$

- Higher proof order with better conditioning

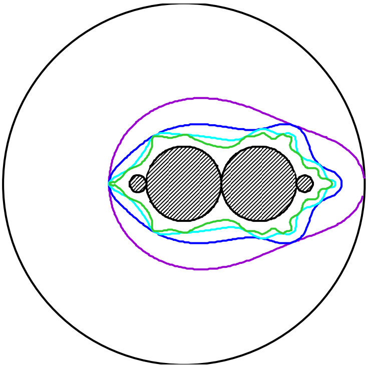
Outer Bounds

Assume

$\lambda \notin \{|\lambda| \leq 1\} \cup \{|\lambda - 2| \leq 1\}$

$V(z_k) = |z_k|^2$ increases

$$V(z_k) \leq V(z_{k+1})$$
$$\Leftrightarrow 1 \leq |\lambda||(1 - z_k)|$$
$$\Leftarrow |z_k| - 1 \geq \frac{1}{|\lambda|}$$
$$\Leftrightarrow |z_k| \geq \frac{1}{|\lambda|} + 1$$



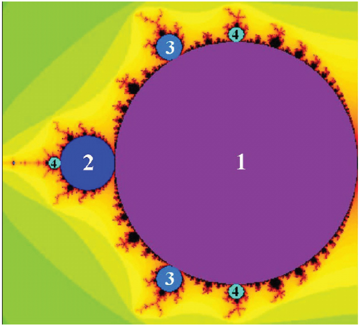
Example (First iteration $z_0 = \frac{1}{2}$)

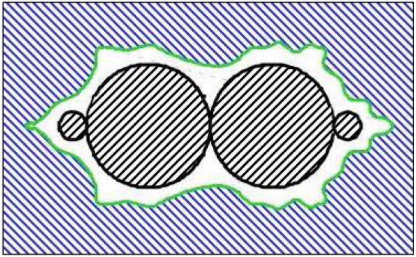
$$\left|\frac{\lambda}{4}\right| \geq \frac{1}{|\lambda|} + 1$$
$$\Leftrightarrow |\lambda|^2 + 4|\lambda| - 4 \geq 0$$

Fragility in the Mandelbrot Set

What is easy

- Regional (‘Global’) proofs for the cyclic regions (in both z and λ).
- Proofs for the 2 period lobes are linearized z space (‘global’ in λ).
- Outer bounds for the set.
- The fragility of the unresolved points is easily established.
  - “ White region is fragile” is a robust theorem and has a short proof.
  - Membership in white region is fragile and has complex proof.





Summary

Summary

How might this help with organized complexity and robust yet fragile?

- Long proofs indicate a fragility.
  - Either a true fragility (a useful answer) or artifact of the model (which must then be rectified).
- This example is much simpler than general dynamical systems where we cannot visualize things.
- SOS methods and tools (SOSTOOLS) give general purpose method to generate short proofs for Mandelbrot set and other dynamical systems.