Krishna Shankar
21st Jan 2013

CDS 110b – Example

Necessary Conditions VIA Calculus of Variations

Problem: Minimize $\int_0^T L(x,u,t)\,dt + V(x(T))$

subject to $\dot{x} = f(x,u)$

$T = T_0$ = constant.

"Augmented Cost function" – J :

$$J(x,u,t,T,\lambda) = V(x(T)) + \int_0^T \left[L(x,u,t) + \lambda^T (f(x,u) - \dot{x}) \right] dt$$

We want to find the stationary value of J (a minimum).
Recall from calculus that the necessary condition for a function $f : \mathbb{R}^n \to \mathbb{R}$ to have a minimum at $\vec{x}_0$ is

$$\nabla_x f \Big|_{x_0} = 0$$

Along the same lines, we say $\delta J \Big|_{x^*, u^*, \lambda^*, \dot{x}^*} = 0$

$$\delta J = \frac{\partial V}{\partial x}\Big|_{x_f} \delta x_f + \int_0^T \Big[\frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial u} \delta u + (f - \dot{x}) \delta \lambda + \lambda^T \Big(\frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial u} \delta u - \delta \dot{x} \Big) \Big] dt$$

Let $H = L(x,u,t) + \lambda^T f(x,u)$ ("Control Hamiltonian")

Note: we denote $x(T)$ as x_f

Krishna Shankar

We can rewrite δJ as

$$\delta J = \left(\frac{\partial V}{\partial x} - \lambda^T\right)\bigg|_{x_f} \delta x_f + \Big[L + \lambda^T(f - \dot{x})\Big]_T \delta T$$
$$+ \int_0^T \left[\frac{\partial H}{\partial x}\delta x + \frac{\partial H}{\partial u}\delta u + (f-\dot{x})^T \delta\lambda + \underbrace{\lambda^T \delta\dot{x}}_{*}\right] dt$$

we can rewrite * as

$$\int_0^T \lambda^T \left(\frac{d}{dt}(\delta x)\right) dt \xrightarrow{\text{integration by parts}} = \lambda^T \delta x \Big|_0^{T_f} - \int_0^T \dot{\lambda}^T \delta x \, dt$$

$$= \lambda(T)^T \delta x_F - \int_0^T \dot{\lambda}^T \delta x \, dt$$

$$\Rightarrow \delta J = \left(\frac{\partial V}{\partial x} - \lambda^T\right)\delta x_F + \Big[L + \lambda^T(f - \cancel{\dot{x}}) + \cancel{\lambda^T \dot{x}}\Big]_T \delta T$$
$$+ \int_0^T \left[\left(\frac{\partial H}{\partial x} + \dot{\lambda}^T\right)\delta x + \frac{\partial H}{\partial u}\delta u + (f - \dot{x})^T \delta\lambda\right] dt$$

In this case, $\delta T = 0$ since T is constant/prescribed.

We set $\delta J = 0$ — this means that the coefficients of variations (δx_F, δx, δu, $\delta\lambda$) must all be identically zero (since variations are arbitrary, non-zero!)

Let's go through the r.h.s of $\delta J = 0$ term by term:

$$\left(\frac{\partial V}{\partial x} - \lambda^T\right)\Big|_T \delta x_f = 0 \implies \boxed{\frac{\partial V}{\partial x}\Big|_T = \lambda^T(T)}$$

$$\left[L + \lambda^T f\right]_T \delta T = 0 \implies \text{no info, since } \delta T = 0$$

$$\int_0^T \left(\frac{\partial H}{\partial x} + \dot{\lambda}^T\right) \delta x\, dt = 0 \implies \boxed{\dot{\lambda} = -\left(\frac{\partial H}{\partial x}\right)^T}$$

$$\int_0^T \frac{\partial H}{\partial u} \delta u\, dt = 0 \implies \boxed{\frac{\partial H}{\partial u} = 0}$$

$$\int_0^T (f - \dot{x})^T \delta \lambda\, dt \implies \dot{x} = f(x,u), \text{ a.k.a. } \boxed{\frac{\partial H}{\partial \lambda} = \dot{x}}$$

Notice that $\frac{d}{dt}\left(H(x^*, \lambda^*, u^*)\right) = \frac{\partial H}{\partial x}\dot{x} + \cancel{\frac{\partial H}{\partial u}}\dot{u} + \frac{\partial H}{\partial \lambda}\dot{\lambda}$

$$= -\dot{\lambda}\dot{x} + \dot{x}\dot{\lambda}$$

$$= 0$$

$$\implies \boxed{H(x^*, \lambda^*, u^*) = \text{constant}}$$