Transfer functions, block diagram algebra, and Bode plots

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11/05/15

Going back and forth between the time and the frequency domain (1)

- Transfer functions exist only for linear systems! If the system is nonlinear and you want to compute its transfer function, you must linearize it first.

- How do we go back and forth between the state space and the frequency domain?

Going back and forth between the time and the frequency domain (2)

- The key fact is that the linear system $dx/dt = Ax + Bu$, $y = Cx + Du$ in the state space has the transfer function $G(s) = C(sI - A)^{-1}B + D$.

- Another possibility is to use the table of Laplace transform functions from lecture:

  $dx_1 / dt = x_2 \quad \Rightarrow sX_1(s) = X_2(s)$

  $dx_2 / dt = 3x_1 + 4x_2 \quad \Rightarrow sX_2(s) = 3X_1(s) + 4X_2(s)$

- Then solve for the Laplace transforms X_1 and X_2.

- Similarly $\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = u \Rightarrow s^2Y(s) + 2s\zeta\omega_nY(s) + \omega_n^2Y(s)$

$= U(s) \Rightarrow G_{yu}(s) = Y(s) / U(s) = 1 / (s^2 + 2\zeta\omega_ns + \omega_n^2)$

Transfer function properties and MATLAB code (1)

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$G(s) = \frac{n(s)}{d(s)}$$
$$d(s) = \det(sI - A)$$

- Roots of *d*(*s*) are called *poles* of *G*(*s*)
- Roots of *n*(*s*) are called *zeros* of *G*(*s*)

Poles of G(s) determine the stability of the (closed loop) system

- $G(s) = C(sI - A)^{-1}B + D$ (D is usually 0).
- When **deg n < deg d**, the transfer function is called *proper*
- **G_{yu} = Y(s) / U(s)** –> not the same as U(s) / Y(s), make sure you take the correct ratio of Laplace transforms
- When presented with a transfer function, the information is encoded in the **poles and zeros**
- The **poles** are the **roots of d(s)** and the **zeros** are **the roots of n(s)**

Transfer function properties and MATLAB code (2)

- The **poles** of the transfer function determine the **stability** of the **closed loop system** -> more about that later

- The **zeros** of the transfer function hide certain **inputs** and can mask fragility -> more about that later too

- When poles and zeros cancel each other, the system is poorly designed, though algebraically valid.

- MATLAB commands:

1. function **tf**:

- num = [1 2]; den = [1 2 3]; tf(num, dem) -> (s + 2) / (s^2 + 2s + 3)

2. function **bode** plots the frequency plot:

- bode(tf(num, dem))
- **The magnitude is plotted in dB by default!** You can convert the magnitude from absolute units to decibels using: **magnitude_db = 20*log10(magnitude)**.
- Many other functions in MATLAB for studying the frequency domain. If you can think about it, it probably exists.

Block diagram algebra General principles of block reduction

- You can reduce blocks according to the rules:

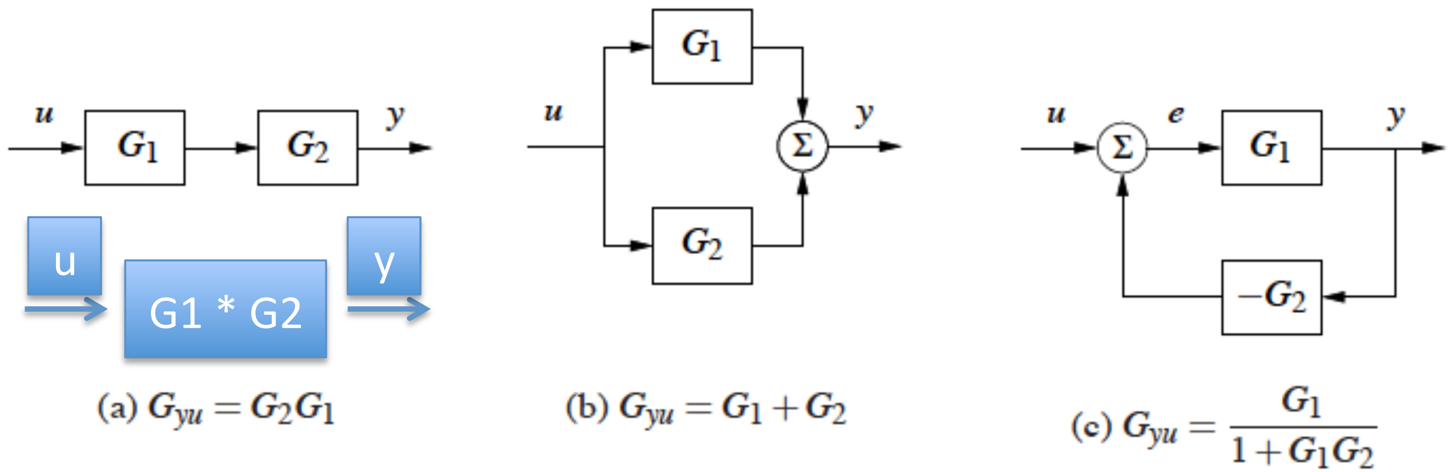
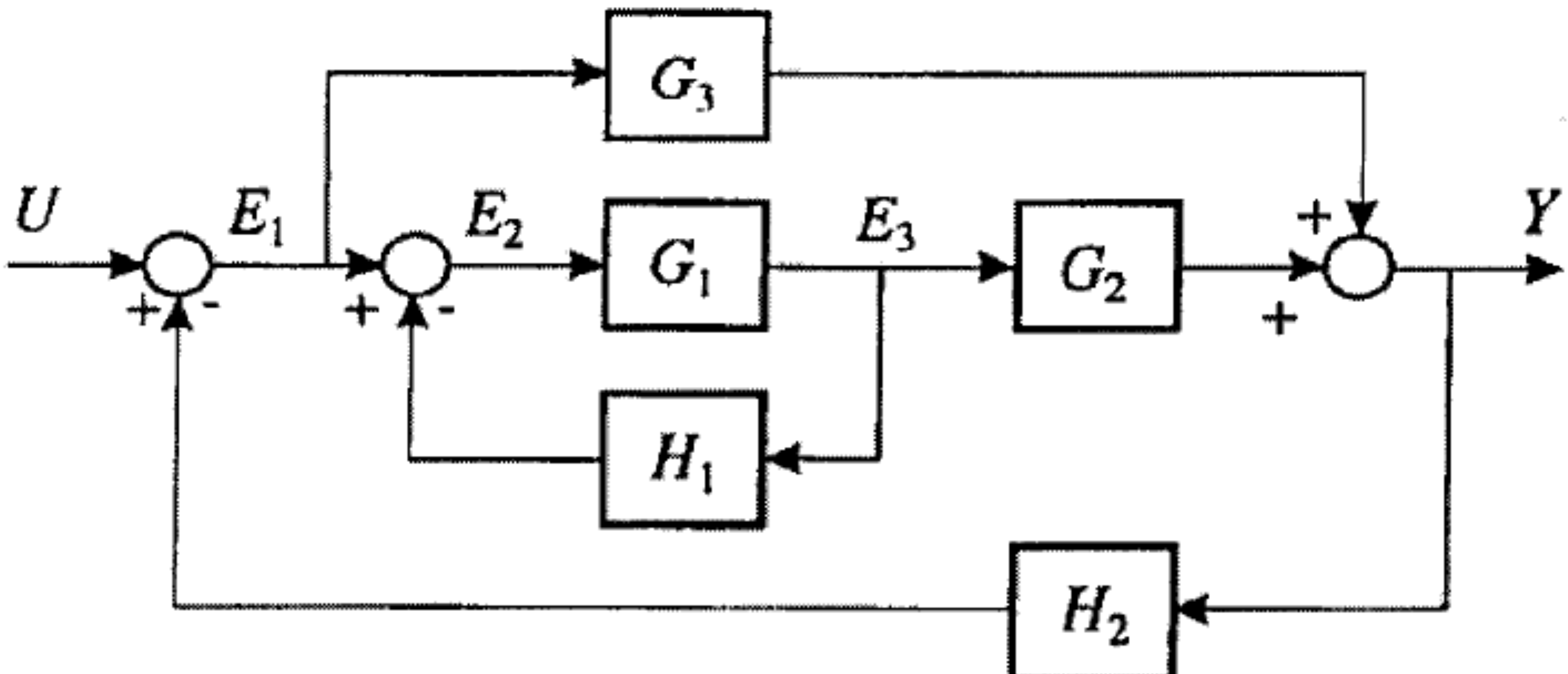


Figure 8.6: Interconnections of linear systems. Series (a), parallel (b) and feedback (c) connections are shown. The transfer functions for the composite systems can be derived by algebraic manipulations assuming exponential functions for all signals.

- There are other block manipulation rules

Block diagram algebra
Example system

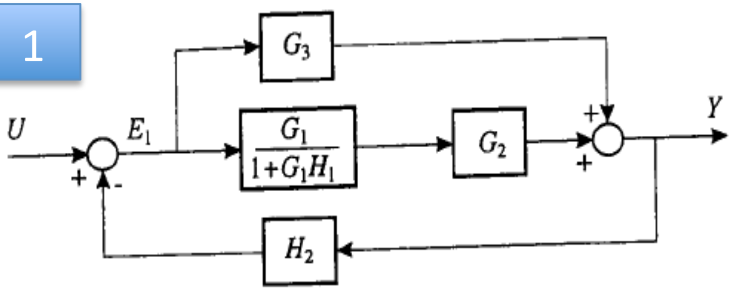
- Reduce the blocks to compute the input-output function for the system:



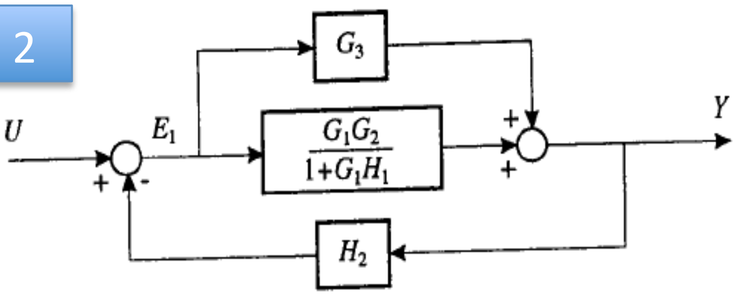
Block diagram algebra
Example system solution

- Proceed in the 4 steps below, working on the smaller loops first.

1



2



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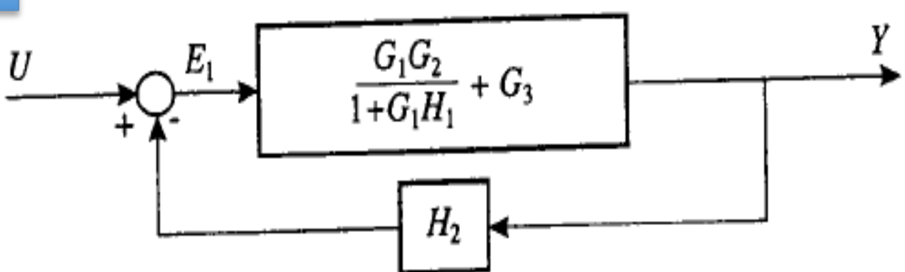

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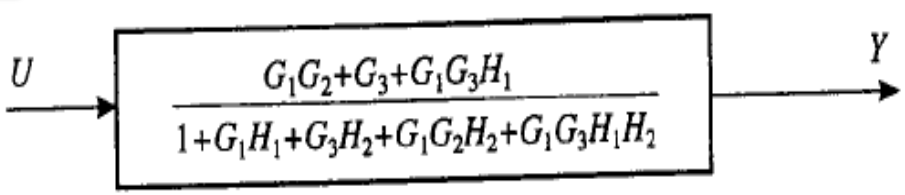
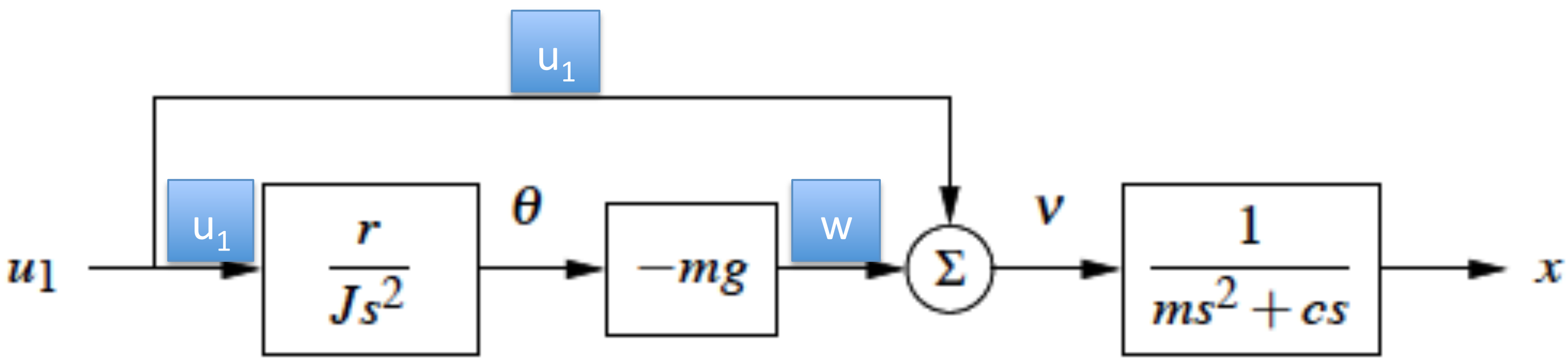


Block diagram algebra
Working with signals (1)

- Label the arrows on the block diagram with signal names and then derive the algebraic relationships between the Laplace transforms.



- $w = \theta * (-mg)$

  $x = v * 1 / (ms^2 + cs)$
- $\theta = u_1 * r / (Js^2)$
- $v = w + u_1$

Block diagram algebra
Working with signals (2)

- Let's compute the transfer functions **1. G_{xu1}** and **2. $G_{\theta v}$**.

1. We use: $x = v / (ms^2 + cs)$

$= (w + u_1) / (ms^2 + cs)$

$= (\theta * (-mg) + u_1) / (ms^2 + cs)$

$= u_1 * (1 + r/ (Js^2) * (-mg)) / (ms^2 + cs)$

$= u_1 * (Js^2 - mgr / (Jm\ s^4 + cJ\ s^3))$

Hence, $G_{xu1} = x / u_1 = (Js^2 - mgr) / (Jm\ s^4 + cJ\ s^3)$. It has a triple pole at 0. What value does the forth pole have? What are the values of the zeros?

2. We use: $v = w + u_1$

$= \theta * (-mg) + \theta / r * Js^2$

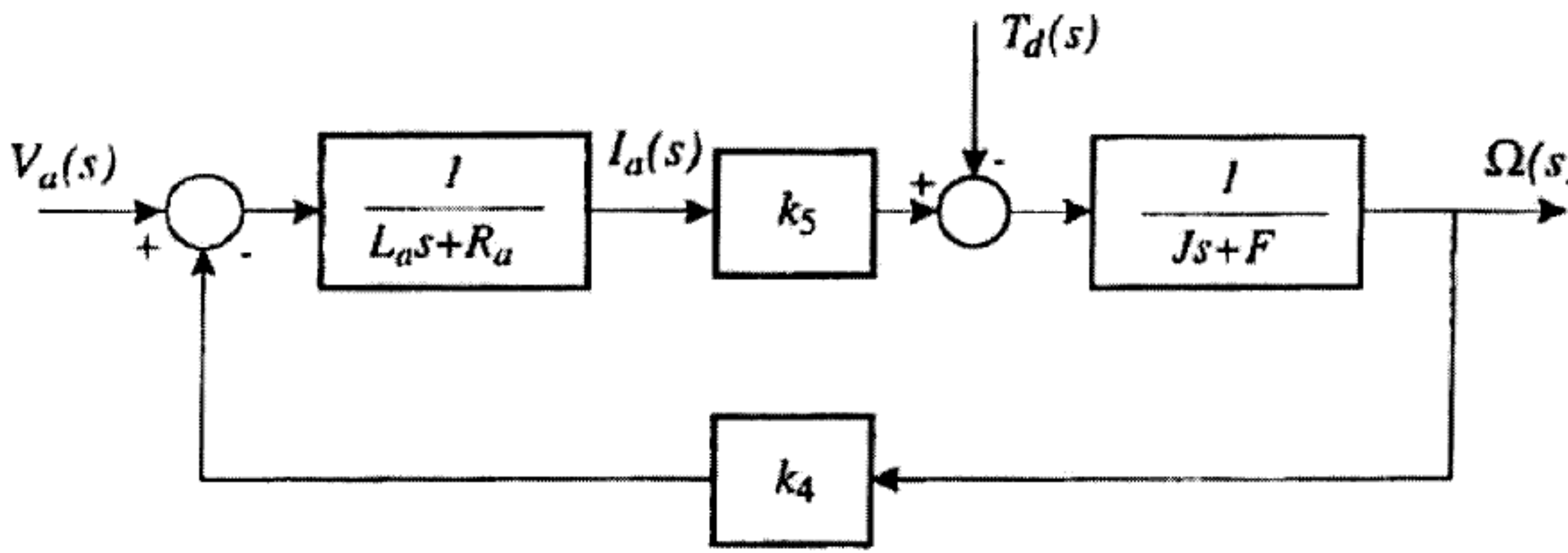
$= \theta * (-mg + J/r * s^2)$

Hence, $G_{\theta v} = \theta / v = 1 / (-mg + J/r * s^2)$. What are the values of the poles of this transfer function?

Main idea: Algebraic manipulation allows you to compute transfer functions between any signals.

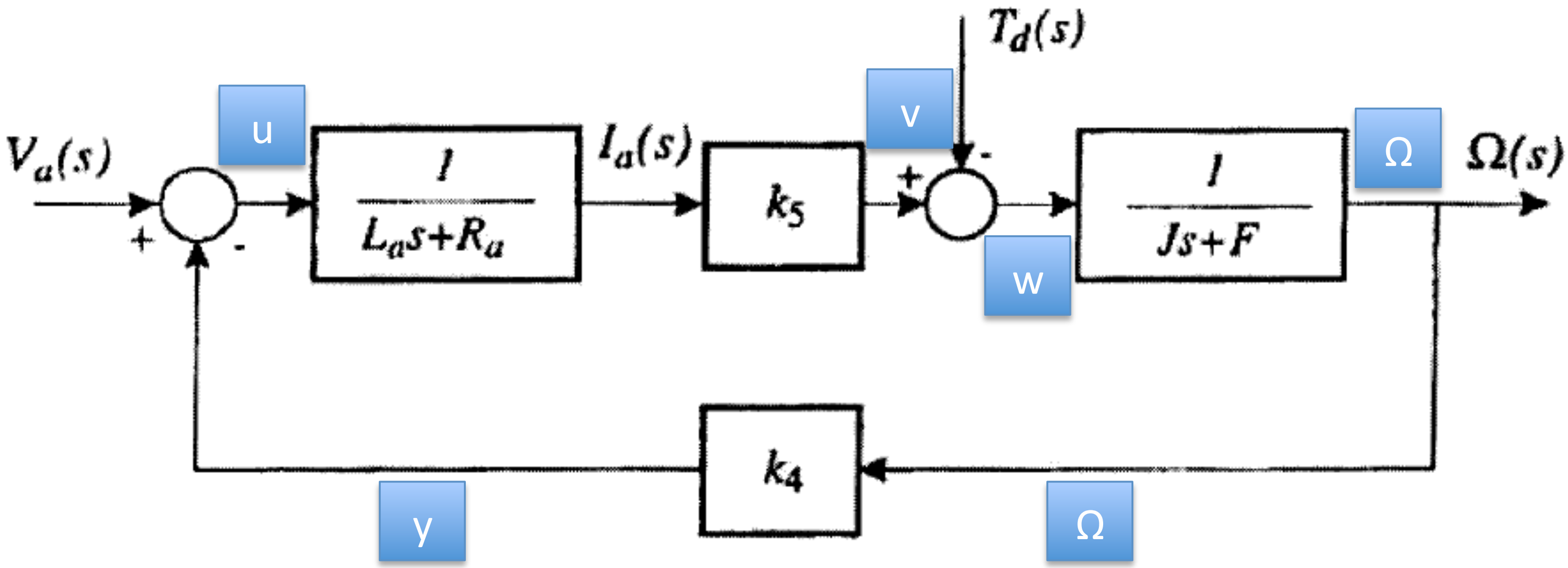
Block diagram algebra Example system

- Find the transfer function G(s) from $T_d(s)$ to Ω(s) and the transfer function H(s) from $V_a(s)$ to Ω(s). Simplify your answer as much as possible.



Block diagram algebra Solution (1)

- We first label all the signals:



Block diagram algebra Solution (2)

- We write down the algebraic relations between signals: $u = V_a - y$, $I_a = u / (L_a s + R_a)$, $v = k_5 I_a$, $w = v - T_d$, $\Omega = w / (Js + F)$, $y = k_4 \Omega$.

- We ignore the input V_a when we compute the transfer function $G(s) = \Omega(s) / T_d(s) = (L_a s + R_a) / (JL_a s^2 + (R_a J + FL_a)s + FR_a + k_4 k_5)$.

- We ignore the disturbance T_d when we compute the transfer function $H(s) = \Omega(s) / V_a(s) = k_5 / (JL_a s^2 + (FL_a + JR_a)s + FR_a + k_4 k_5)$.

Bode plots of transfer functions General approach (1)

- The magnitude of the transfer function $|G(j\omega)|$ is plotted on a log-log plot

- The phase $<G(j\omega)$ is plotted on a log-linear plot

- When we draw bode plots by hand, it's always an approximation

- The magnitude plot starts at $|G(0)|$

- The magnitude plot of a transfer function can always be broken up in a sum due to the fact that $\log|G(j\omega)|$ becomes a sum of logarithms

- What are the parts that you should you break it up in?

Bode plots of transfer functions
General approach (2)

- **How do you break up a complicated transfer function into parts you can actually plot?**

- The transfer function $G(s) = (s + 1) / (s^3 + 4s)$ has a zero at $s = -1$ and three poles at $s = 0$, $s = -2j$, $s = 2j$.

- You break it up as $G(j\omega) = (s + 1) * 1/s * 1/((s - 2j)*(s + 2j))$ =>
  $\log|G(j\omega)| = \log|j\omega + 1| - \log|j\omega| - \log|(j\omega)^2 + 4|$
  $<G(j\omega) = <(j\omega + 1) - <(j\omega) - <(j\omega - 2j) - <(j\omega + 2j)$

- In general, you separate the numerator and denominator in products of:
- Constants, Poles at the origin, Zeros at the origin, Real Poles, Real Zeros
- Complex conjugate poles
- Complex conjugate zeros

- Then you plot them separately and the magnitude and phase plots will be the sum of their magnitudes.

Bode plots of transfer functions General approach (3)

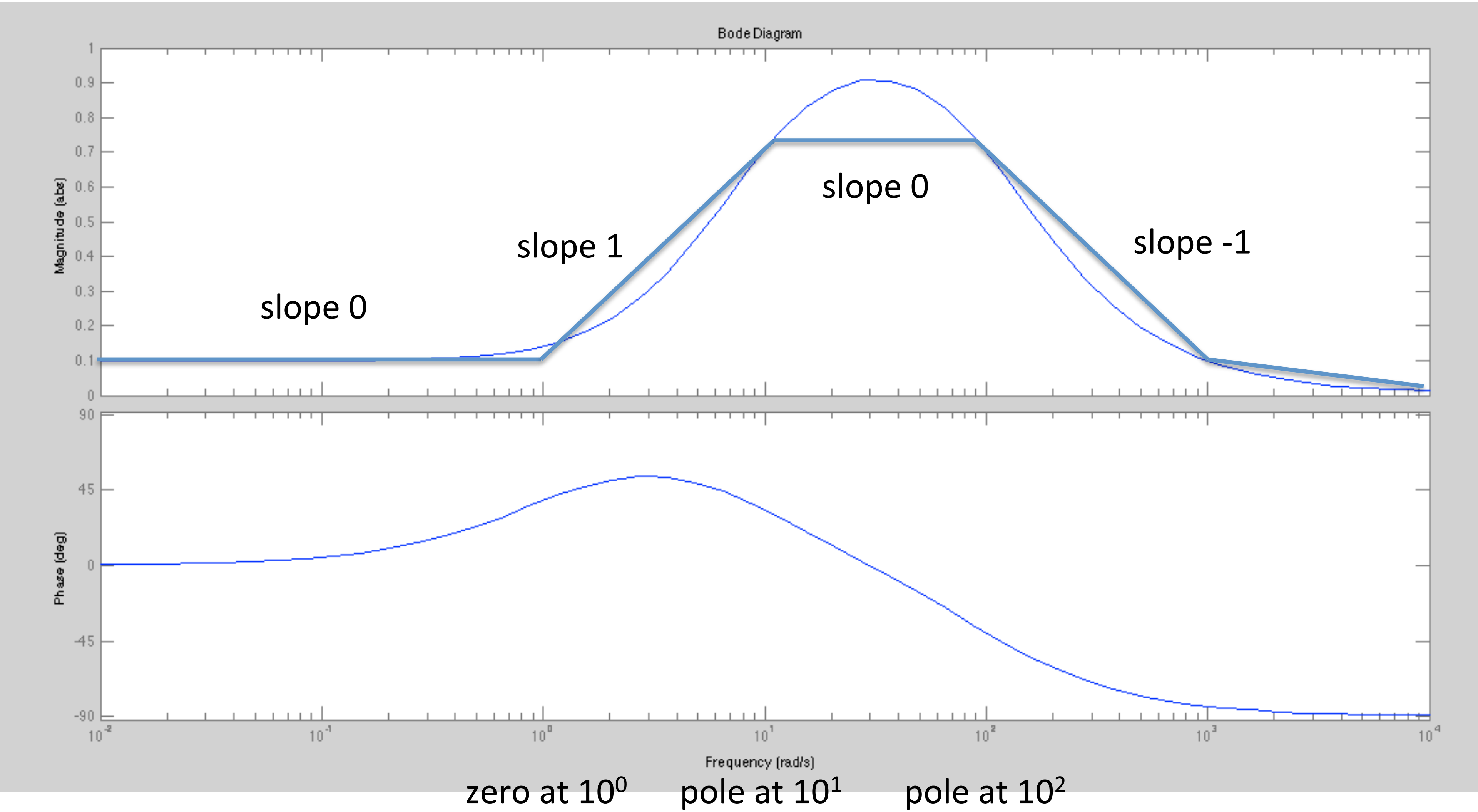
General guidelines for drawing bode plots:

- A simple zero with negative real part pushes magnitude up with slope +1 and pushes phase up by (up to) +90 degrees -> transfer function s+a, a>0
- A simple pole with negative real part pushes magnitude down with slope -1 phase down by (up to) -90 degrees -> transfer function 1/(s+a), a>0
- A double pole with roots with negative real parts pushes magnitude down with slope -2 and pushes phase down by (up to) -180 degrees -> $1/(s^2 + 2s+2)$; the analogous double zero pushes phase up by (up to) +180 degrees
- The (up to) accounts for the existence of other poles and zeros close by that might influence if you actually get to +90 degrees (you might only get up to +80 before a nearby pole or zero forces you to come down).
- The influence of a pole/zero of value v on the phase is visible on the interval (v/10, 10v). It might push the phase up or down.
- Don't forget to account for (ω ~ 0) and (ω -> infinity) in your bode plot.

Bode plots
A complicated function (1)

- $H(s) = (100s + 100) / (s^2 + 110s + 1000)$.
- So $H(s) = 100 * (s + 1) / ((s + 10) * (s + 100))$
- It has a zero at $s = -1$ and two poles at $s = -10$ and $s = -100$.
- Let's consider the magnitude plot first.
- It starts with $|H(0)| = 0.1$.
- We go up at frequency 10^0 with slope 1, then stay close to slope 0 around 10^1 and go down at 10^2 with slope -1. At very large frequency, we go to 0.
- The phase plot starts at 0 degrees. The zero causes it to go up 90 degrees (between 1/10 and 10). The poles bring it down 90 degrees each (the first between 1 and 100, the second between 10 and 1000).

Bode plots
A complicated function (2)



Bode plots
A pair of complex conjugate poles (1)

$$H(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} = \frac{1}{\left(\frac{s}{\omega_0}\right)^2 + 2\zeta\left(\frac{s}{\omega_0}\right) + 1}, \quad \text{with } 0 < \zeta < 1$$

The magnitude of H(jω)

$$|H(j\omega)| = \left|\frac{1}{\left(\frac{j\omega}{\omega_0}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_0}\right) + 1}\right| = \left|\frac{1}{-\left(\frac{\omega}{\omega_0}\right)^2 + j2\zeta\left(\frac{\omega}{\omega_0}\right) + 1}\right| = \left|\frac{1}{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right) + j\left(2\zeta\left(\frac{\omega}{\omega_0}\right)\right)}\right|$$

$$= \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_0}\right)^2}}$$

$$\log_{10}|H(j\omega)| = -\log_{10}\left(\sqrt{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_0}\right)^2}\right)$$

Bode plots
A pair of complex conjugate poles (2)

- There are 3 cases to consider for the approximation:

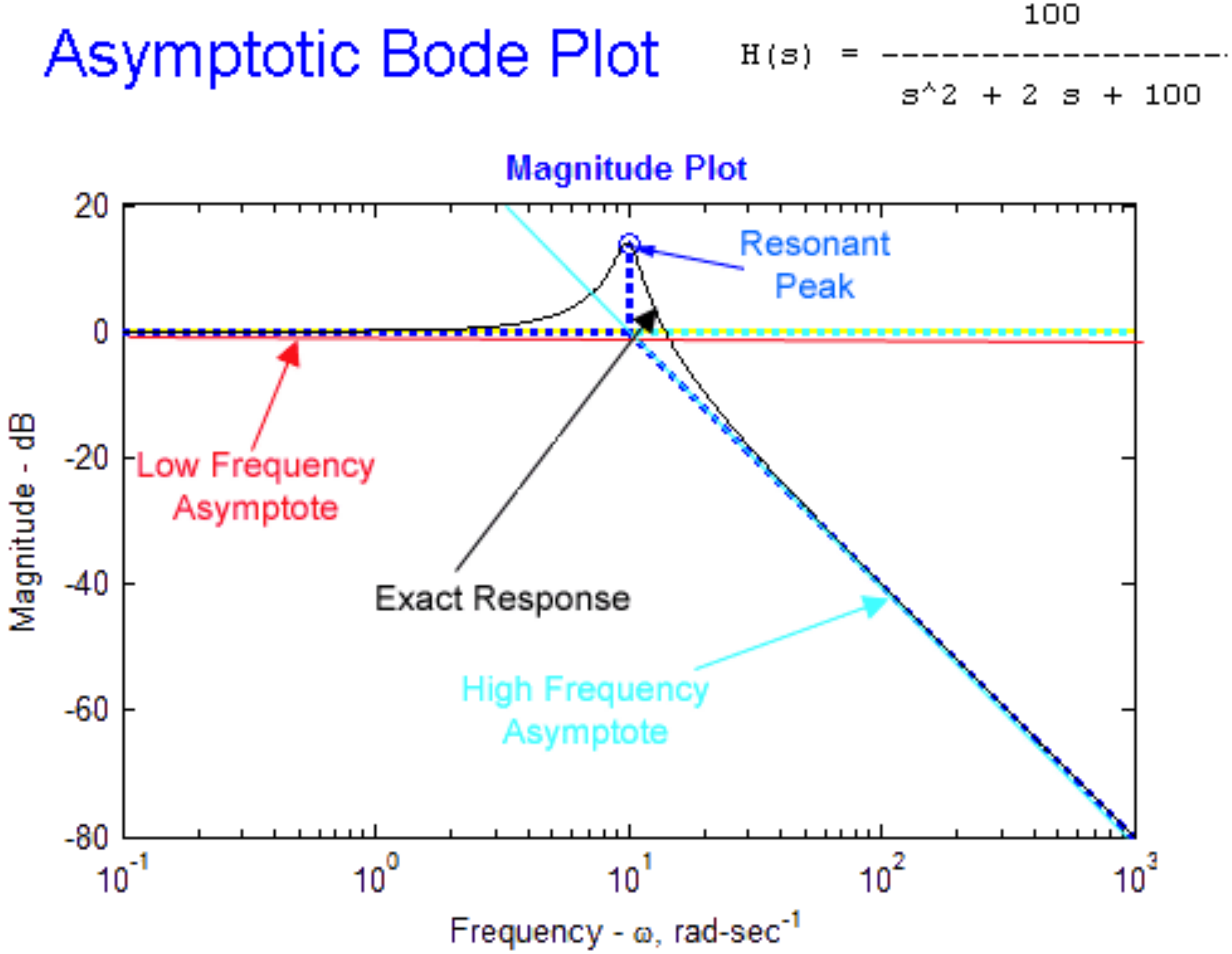
1. When $\boldsymbol{\omega \ll \omega_0}$, then $\log_{10}|H(j\omega)| \sim 0$. This is the low frequency case.
2. When $\boldsymbol{\omega \gg \omega_0}$, then $\log_{10}|H(j\omega)| \sim -\log_{10}((\omega/\omega_0)^2)$. This is the high frequency case.
3. When $\boldsymbol{\omega = \omega_r = \omega_0 * \mathrm{sqrt}(1 - 2\zeta^2) \sim \omega_0}$ for $\zeta < 1/\mathrm{sqrt}(2) \sim 0.5$. This is the resonant frequency of the underdamped second order system.

Bode plots
A pair of complex conjugate poles (3)

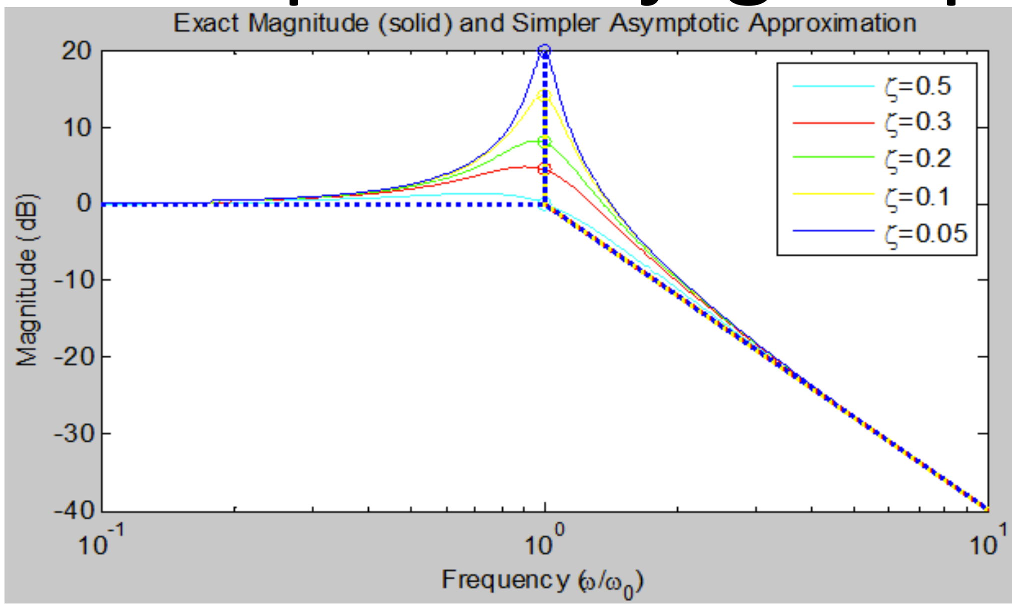
- The peak at the resonant frequency $\boldsymbol{\omega_r}$ has height $\frac{1}{2\zeta\sqrt{1-\zeta^2}}$ ~ 1/(2**ζ**).

- The value of the transfer function at the resonant frequency is $|H(j\omega_r)| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$

Bode plots
A pair of complex conjugate poles (4)



Bode plots
A pair of complex conjugate poles (5)



- Please notice how the peak changes wrt the value of **ζ**.
- For a pair of complex conjugate zeros, at the resonant frequency, there will be a dip and not a peak. The magnitude will be the same as for the peak.
- If the transfer function has the expression $s^2 - 2\zeta\omega_0 s + \omega_0^2$ instead of $s^2 + 2\zeta\omega_0 s + \omega_0^2$, then amplify it by its conjugate to apply the same reasoning.

Bode plots
A pair of complex conjugate plots (6)

- This is how to plot the phase:

Case 1) $\omega \ll \omega_0$. This is the low frequency case. At these frequencies We can write an approximation for the phase of the transfer function

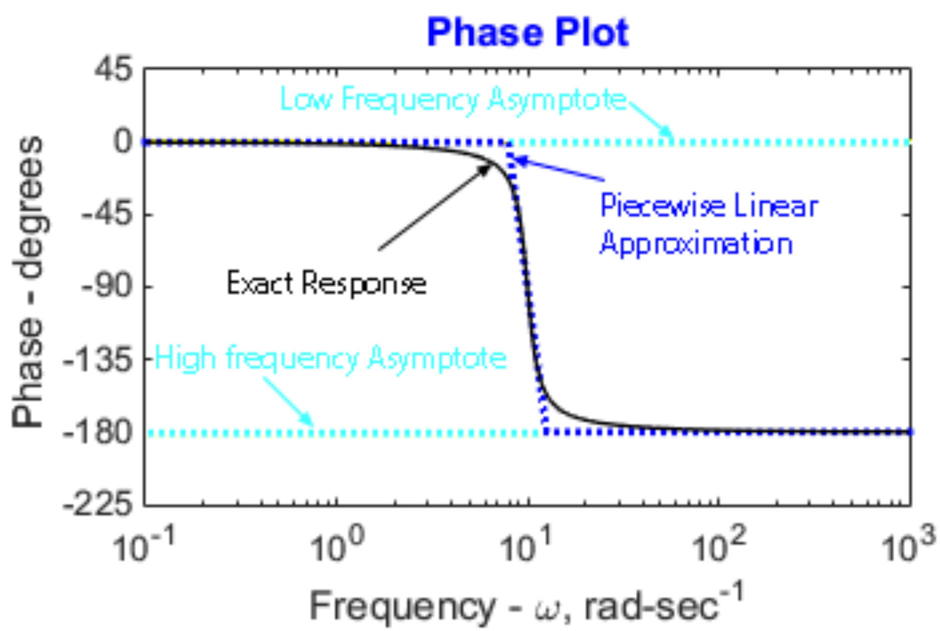
$$\angle H(j\omega) \approx -\arctan(0) = 0° = 0 \text{ radians}$$

Case 2) $\omega \gg \omega_0$. This is the high frequency case. We can write an approximation for the phase of the transfer function

$$\angle H(j\omega) = -180°$$

Case 3) $\omega = \omega_0$. The break frequency. At this frequency

$$\angle H(j\omega) = -90°$$



Other comments about bode plots

- 360 degrees = 0 degrees on the phase portraits
- -270 degrees = 90 degrees on the phase portraits

- The magnitude of transfer functions s-1 and s+1 is the same, but their zeros are in different places: at 1 and respectively -1. On the hand drawn magnitude bode plot, they should look the same. However, the phases should be different. The phase changes by +90 degrees for s-1 and by -90 degrees for s+1.

- The way to make sense of this is that $s+1 = (s^2-1)/(s-1)$. So $j\omega+1 = -(\omega^2+1)/(j\omega-1)$. This function has a constant on the numerator and a pole at value 1 on the denominator. Hence, the effect of a **zero with positive real part on the numerator** is the same as the effect of **a pole with negative real part on the denominator**.

WARNING: Do not actually do this for control design problems because it introduces fragility. This is just to give you intuition for drawing bode plots.

Code to change your axes from dB to absolute value on the magnitude plot in MATLAB

```
h=bodeplot(tf(num, den));
a=getoptions(h); a.MagUnits='abs';
setoptions(h,a);
```

Useful materials

- http://faculty.mu.edu.sa/public/uploads/1415021770.8406Block_Diagram_Reduction_Rules.pdf -> block diagram reduction
- http://lpsa.swarthmore.edu/Bode/Bode.html -> all about bode plots
- http://lpsa.swarthmore.edu/Bode/BodeReviewRules.html -> table of rules for how to plot bode plots

References

- The course textbook “Feedback Systems” by K. J. Astrom and R. M. Murray.
- Midterm for MAE 143B, Prof. M. Krstic, May 1, 2007.