CDS 140a Winter 2014 Homework 7

From MurrayWiki

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ACM 101b/AM 125b/CDS 140a, Winter 2014 Due: 16 Feb 2014 (Wed) @ noon

(PDF)

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. **Perko, Section 3.4, problem 1**: Show that $\gamma(t) = (2\cos 2t, \sin 2t)$ is a periodic solution of the system
$$\dot{x} = -4y + x\left(1 - \frac{x^2}{4} - y^2\right)$$
$$\dot{y} = x + y\left(1 - \frac{x^2}{4} - y^2\right)$$
that lies on the ellipse $(x/2)^2 + y^2 = 1$ (i.e., $\gamma(t)$ represents a cycle Γ of this system). Then use the corollary to Theorem 2 in Section 3.4 to show that Γ is a stable limit cycle.

2. **Perko, Section 3.4, problem 3a**: Solve the linear system
$$\dot{x} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$
and show that any at point $(x_0, 0)$ on the x-axis, the Poincare map for the focus at the origin is given by $P(x_0) = x_0 \exp(2\pi a / |b|)$. For $d(x) = P(x) - x$, compute $d'(0)$ and show that $d(-x) = -d(x)$.

3. **Perko, Section 3.5, problem 1**: Show that the nonlinear system
$$\dot{x} = -y + xz^2$$
$$\dot{y} = x + yz^2$$
$$\dot{z} = -z(x^2 + y^2)$$
has a periodic orbit $\gamma(t) = (\cos t, \sin t, 0)$. Find the linearization of this system about $\gamma(t)$, the fundamental matrix $\Phi(t)$ for the autonomous system that satisfies $\Phi(0) = I$, and the characteristic exponents and multipliers of $\gamma(t)$. What are the dimensions of the stable, unstable and center manifolds of $\gamma(t)$?

4. **Perko, Section 3.5, problem 5a**: Let $\Phi(t)$ be the fundamental matrix for $\dot{x} = A(t)x$ satisfying $\Phi(0) = I$. Use Liouville's theorem, which states that
$$\det \Phi(t) = \exp \int_0^t \mathrm{trace} A(s) ds,$$
to show that if $m_j = e^{\lambda_j T}, j = 1, \ldots, n$ are the characteristic multipliers of $\gamma(t)$ then

$$\sum_{j=1}^{n} m_j = \text{trace}\,\Phi(T)$$

and

$$\prod_{j=1}^{n} m_j = \exp \int_0^T \text{trace}\,A(t)\,dt.$$

5. **Perko, Section 3.9, problem 4a**: Show that the limit cycle of the van der Pol equation

$$\dot{x} = y + x - x^3/3$$
$$\dot{y} = -x$$

must cross the vertical lines $x = \pm 1$.