Lecture Summary: Networked Control with Delayed and/or Lost Packets

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Process and channel description: Consider the LQG framework for the set-up shown in Figure 1. The process and sensor equations are

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + w(k) \\ y(k) &= Cx(k) + v(k), \end{aligned}$$

and the cost function is

$$J_{LQG} = E\left[\sum_{k=1}^{K}\left(x^T(k)Qx(k) + u^T(k)Ru(k)\right) + x^T(K+1)P(K+1)x(K+1)\right],$$

where the expectation at time k is taken with respect to all the uncertainty in the system and $P(K+1)$, Q and R are all positive definite. The random variables $w(k)$, $v(k)$ and $x(0)$ are Gaussian with

$$E\left[\begin{bmatrix} w(k) \\ v(k) \\ x(0) \end{bmatrix}\begin{bmatrix} w(j) & v(j) & x(0) \end{bmatrix}\right] = \begin{bmatrix} R_w\delta_{jk} & 0 & 0 \\ 0 & R_v\delta_{jk} & 0 \\ 0 & 0 & \Pi(0) \end{bmatrix}.$$

The pair (A,B) is controllable and (A,C) is observable. A minimum mean squared error (MMSE) estimation problem can also be posed by assuming that the process evolves without $u(k)$ being applied.

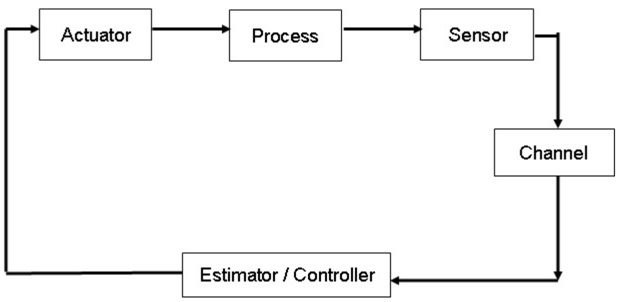

Figure 1:

The channel introduces random delays and loss. The delay is equal to m time steps with a probability p_m for $m = 0, 1, \cdots, d_{\max}$. Moreover, the data is erased with a probability $p_{m+1} = 1 - \sum_{m=0}^{d_{\max}} p_m$. The receiver is aware of the delay or the erasure. Other models of delays such as Maximum Allowable Transmit Interval (MATI) based models, or the probabilities p_m being time-varying (e.g., as a Markov chain) can also be considered. Quantization effects and channel noise are ignored.

One-block design In the one-block design problem the sensor transmits $y(k)$ at time k, and the estimator or the controller has access to all the successfully transmitted measurements. The optimal estimator or controller

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can be designed and analyzed using the MJLS theory. Define

$$z(k)=\begin{bmatrix} x(k)\\ x(k-1)\\ x(k-2)\\ \vdots\\ x(k-d_{\max})\end{bmatrix}\qquad \bar{A}=\begin{bmatrix} A & 0 & \cdots & 0\\ I & 0 & \cdots & 0\\ 0 & I & 0 & \cdots\\ 0 & 0 & \ddots & 0\\ 0 & \cdots & I & 0\end{bmatrix}\qquad \bar{M}=\begin{bmatrix} I\\ 0\\ 0\\ \vdots\\ 0\end{bmatrix}\qquad \bar{v}(k)=\begin{bmatrix} v(k)\\ v(k-1)\\ v(k-2)\\ \vdots\\ v(k-d_{\max})\end{bmatrix}.$$

Then the system can be described by the MJLS

$$\begin{aligned} z(k+1) &= \bar{A}z(k)+\bar{M}Bu(k)+\bar{M}w(k)\\ \bar{y}(k) &= C_{r(k)}z(k)+\bar{v}(k), \end{aligned}$$

where the matrix $C_{r(k)}$ is an block matrix with $N(d_{\max}+1)$ blocks, each of size tn where $x(k)\in\mathbf{R}^n$ and $y(k)\in\mathbf{R}^t$. $C_{r(k)}$ varies according to the stochastic mode $r(k)\in\{0,1,2,\cdots,d_{\max}+1\}$. For $r(k)=0,\cdots,d_{\max}$, the mode corresponds to $r(k)$ time steps of delay occurring. The $(r(k)+1,r(k)+1)$ block of the matrix $C_{r(k)}$ is equal to C, and every other block is equal to a zero matrix. For $r(k)=d_{\max}+1$, the mode corresponds to packet erasure, and $C_{r(k)}$ is equal to a zero matrix.

The optimal estimator and controller can now be designed and analyzed. As an example, the separation principle holds, and the optimal estimator is given by a time-varying Kalman filter. Some rather surprising results may result, e.g., the necessary condition ($p_{m+1}\rho(A)^2<1$), and sufficient condition (as an LMI) for stability are independent of any finite delay.

If there is also a channel between the controller and the actuator, then an acknowledgement is required for any successful packet transmission so that the controller is aware of the Markov mode $r(k)$ at every time k. This is sometimes called the TCP-like case. The UDP-like case when acknowledgements are not available is much more difficult to solve in general.

Two block design In the two-block design problem, an encoder collocated with the sensor and the decoder located at the estimator / controller have to be designed. The encoder calculates and transmits a vector $s(k)=f\left(k,\{y(j)\}_{j=0}^{k}\right)$. The only constraints on the encoder are that the transmitted vector is some causal (possibly time-varying) function of the measurements available to the encoder until time k and that the dimension of the vector is finite. The design of the optimal encoder and decoder proceeds in the following steps. To begin with, consider that the channel is an analog erasure, and every packet that is delivered does not suffer any delay.

- The performance is upper-bounded by an encoder $\mathcal{E}_{opt}$ that transmits all measurements $y(0)$, $y(1)$, $\cdots$, $y(k)$ to the decoder at every time step k. This encoder is not in the class of valid encoders since it transmits a vector with increasing dimension as k increases. With $\mathcal{E}_{opt}$ as the encoder, the decoder has access to an information set of the form $I^{\max}(k)=\{y(0),y(1),\cdots,y(t_s(k))\}$, where $t_s(k)\le k-1$ is the maximal time such that the erasures did not allow any information transmitted by the encoder after time $t_s(k)$ to reach the decoder.
- For algorithm $\mathcal{E}_{opt}$, a *separation principle* holds. The optimal control input is given by

$$u(k)=\hat{u}_{LQ}\left(k|I^{\max}(k),\{u(j)\}_{j=0}^{k-1}\right),$$

  where $u_{LQ}(k)$ is the optimal LQ control law and $\hat{\alpha}(k|\beta(k))$ is the minimum mean squared error (MMSE) estimate of the random variable $\alpha(k)$ based on the information $\beta(k)$. Thus, any algorithm that ensures that the decoder has access to $\hat{u}_{LQ}\left(k|I^{\max}(k),\{u(j)\}_{j=0}^{k-1}\right)$ (or, in turn, $\hat{x}_{LQ}\left(k|I^{\max}(k),\{u(j)\}_{j=0}^{k-1}\right)$) will achieve the same (optimal) performance as $\mathcal{E}_{opt}$.
- Since the optimal MMSE estimate of $x(k)$ is linear in the effects of the maximal information set and the previous control inputs, the encoder need only transmit a quantity that depends only on the measurements. The quantity $\hat{x}_{LQ}\left(k|I^{\max}(k),\{u(j)\}_{j=0}^{k-1}\right)$ can be calculated as

$$\hat{x}_{LQ}\left(k|I^{\max}(k),\{u(j)\}_{j=0}^{k-1}\right)=\bar{x}_{LQ}\left(k|I^{\max}(k)\right)+\psi(k),$$

where $\bar{x}_{LQ}(k|I^{\max}(k))$ depends only on $I^{\max}(k)$ but not on the control inputs and $\psi(k)$ depends only on the control inputs $\{u(j)\}_{j=0}^{k-1}$. Further both $\bar{x}_{LQ}(k|I^{\max}(k))$ and $\psi(k)$ can be calculated recursively using the following equations:

$$\begin{aligned}\bar{x}_{LQ}(k|I^{\max}(k)) &= A^{k-t_s(k)-1}\check{x}(t_s(k)+1|t_s(k))\\ \psi(k) &= A^{k-t_s(k)-1}\check{\psi}(t_s(k)+1)+\sum_{i=0}^{k-t_s(k)-2}A^iBu(k-i-1),\end{aligned}$$

where $\check{x}(j+1|j)$ evolves as

$$\begin{aligned}M^{-1}(j|j) &= M^{-1}(j|j-1)+C^TR_v^{-1}C\\ M^{-1}(j|j)\check{x}(j|j) &= M^{-1}(j|j-1)\check{x}(j|j-1)+C^TR_v^{-1}y(j)\\ M(j|j-1) &= AM(j-1|j-1)A^T+R_w\\ \check{x}(j|j-1) &= A\check{x}(j-1|j-1),\end{aligned}$$

with the initial conditions $\check{x}(0|-1)=0$ and $M(0|-1)=\Pi(0)$, and $\check{\psi}(j)$ evolves as

$$\begin{aligned}\check{\psi}(j) &= Bu(j-1)+\Gamma(j-1)\check{\psi}(j-1)\\ \Gamma(j) &= AM^{-1}(j-1|j-1)M(j-1|j-2),\end{aligned}$$

with the initial condition $\check{\psi}(0)=0$.

- Thus, the following algorithm $\mathcal{A}_2$ is both valid and optimal. At every time step k, the encoder calculates and transmits the quantity $\check{x}(k|k)$ using the algorithm above. The decoder calculates the quantity $\psi(k)$. If the transmission is successful, the decoder calculates

$$\begin{aligned}\hat{x}_{LQ}\left(k+1|I^{\max}(k+1),\{u(j)\}_{j=0}^{k}\right) &= \bar{x}_{LQ}(k+1|I^{\max}(k+1))+\psi(k)\\ &= A\check{x}(k|k)+\psi(k).\end{aligned}$$

  If the transmission is unsuccessful, the decoder calculates

$$\hat{x}_{LQ}\left(k+1|I^{\max}(k+1),\{u(j)\}_{j=0}^{k}\right) = A^{k-t_s(k)}\bar{x}_{LQ}(k+1|I^{\max}(t_s(k)+1))+\psi(k),$$

  where the quantity $\bar{x}_{LQ}(k+1|I^{\max}(t_s(k)+1))$ is stored in the memory from the last successful transmission (note that only the last successful transmission needs to be stored). This is the solution of the two block design problem.

Remarks: Non-linear Structure Note that the optimal algorithm is non-linear (in particular, it is a switched linear system). This is not unexpected, in view of the non-classical information pattern in the problem.

Boundedness of the Transmitted Quantity The quantity $\check{x}(k|k)$ that the encoder transmits is not the estimate of $x(k)$ (or the state of some hypothetical open loop process) based only on the measurements $y(0), \cdots, y(k)$. In particular if the state of the closed loop system $x(k)$ is stable, the quantity $\check{x}(k|k)$ is bounded. If the closed loop system $x(k)$ is unstable due to high erasure probabilities, $\check{x}(k|k)$ is not bounded. However, the optimality result implies that in this case the system cannot be stabilized by transmitting any other bounded quantity (such as measurements).

Optimality for any Erasure Pattern and the 'Washing Away' Effect The optimality of the algorithm required no assumption about the erasure statistics. The optimality result holds for an arbitrary erasure sequence, and at every time step (not merely in an average sense). Moreover, any successful transmission 'washes away' the effect of the previous erasures in the sense that it ensures that the control input is identical to the case as if all previous transmissions were successful.

Presence of Delays The same algorithm continues to remain optimal even if the channel introduces larger (or even time-varying) delays, as long as there is the provision of a time stamp from the encoder regarding the time it transmits any vector. The decoder uses the packet it receives at any time step only if it was transmitted later than the quantity it has stored from the previous time steps. If this is not true (due to packet re-ordering), the decoder continues to use the quantity stored from previous time steps. Further, if the delays are finite, the stability conditions for the closed loop system unchanged. Infinite delays are equivalent to packet erasures, and can be handled by using the same framework.

Stability and Performance: Using the separation principle, the algorithm can be analyzed using the error of the estimate at the decoder. If the only effect of the channel is data loss with a probability p, then due to the 'washing away' effect of the algorithm, the covariance evolves as

$$P(k+1) = \begin{cases} M(k+1) & \text{with probability } 1-p \\ AP(k)A^T + R_w & \text{with probability } p, \end{cases}$$

where $M(k)$ is the covariance of the error if the state $x(k)$ is estimated using all previous measurements and control inputs. Thus,

$$E[P(k+1)] = (1-p)M(k+1) + pR_w + pAE[P(k)]A^T,$$

where the extra expectation for the error covariance is taken over the erasure process in the channel. Since the system is observable, $M(k)$ converges exponentially to a steady state value $M^\star$. Thus, the necessary and sufficient condition for the convergence of the above discrete algebraic Lyapunov recursion is

$$p\rho(A)^2 < 1,$$

where $\rho(A)$ is the spectral radius of A. Due to the optimality of the algorithm considered above, this condition is necessary for stability of the system with any causal encoding algorithm. This analysis can be generalized to more general erasure models. For example, for a Gilbert-Eliot type channel model, the necessary and sufficient condition for stability is given by

$$q_{00}\rho(A)^2 < 1,$$

where q_{00} is the conditional probability of an erasure event at time $k+1$, provided an erasure occurs at time k. In addition, by calculating the terms $E[P(k)]$ and the LQ control cost of the system with full state information, the performance J_{LQG} can also be calculated through the separation principle. The treatment can also be extended to consider the stability of higher order moments of the estimation error, or the state value. In fact, the entire steady state probability distribution function of the estimation error can be calculated.

Extensions: **Channel between Controller and Actuator:** The encoding algorithm above continues to remain optimal when a channel is present between the controller and the actuator, as long as there is a provision for acknowledgement from actuator to controller for any successful transmission, and the protocol that the actuator follows in case of an erasure is known at the controller. This is because these two assumptions are enough for the separation principle to hold. If no such acknowledgement is available, the control input begins to have a dual effect and the optimal algorithm is still unknown.

The problem of two block design for the controller-actuator channel can also be considered. This design will intimately depend on the information that is assumed to be known at the actuator (e.g., the cost function, the system matrices and so on). Algorithms that optimize the cost function for such information sets are largely unknown. However, necessary and sufficient stability conditions can be calculated.

Presence of a Communication Network or Multiple sensors: In a later lecture, we will consider the case when a communication network instead of a single channel, or multiple sensors are present.

Inclusion of More Communication Effects: Other effects due to communication channels can also be considered. Stability conditions for a channel that introduces both erasures and a bit rate constraint have been identified and are a natural combination of the stability conditions for the analog erasure channel above and the ones for a noiseless digital channel, as considered in a later lecture.