1. **Perko, Section 2.9, problem 3** Use the Lyapunov function $V(x) = x_1^2 + x_2^2 + x_3^2$ to show that the origin is an asymptotically stable equilibrium point of the system

$$
\begin{bmatrix}
-x_2 - x_1 x_2^2 + x_3^2 - x_3^3 \\
x_1 + x_3^3 - x_3^2 \\
-x_1 x_3 - x_3 x_1^2 - x_2 x_3^2 - x_3^5
\end{bmatrix} = Df(0)x
$$

Show that the trajectories of the linearized system $\dot{x} = Df(0)x$ for this problem lie on circles in planes parallel to the $x_1, x_2$ plane; hence, the origin is stable, but not asymptotically stable for the linearized system.

2. Determine the stability of the system

$$
\begin{align*}
\dot{x} & = -y - x^3 \\
\dot{y} & = x^5
\end{align*}
$$

Hint: motivated by the first equation, try a Lyapunov function of the form $V(x, y) = \alpha x^6 + \beta y^2$. Is the origin asymptotically stable? Is the origin globally asymptotically stable?

3. **Definition:** An equilibrium point is **exponentially stable** if $\exists M, \alpha > 0$ and $\epsilon > 0$ such that $\|x(t)\| \leq Me^{-\alpha t} \|x(0)\|$, $\forall \|x(0)\| \leq \epsilon$, $t \geq 0$. Let $\dot{x} = f(x)$ be a dynamical system with an equilibrium point at $x_e = 0$. Show that if there is a function $V(x)$ satisfying $k_1 \|x\|^2 \leq V(x) \leq k_2 \|x\|^2$, $\dot{V}(x) \leq -k_3 \|x\|^2$ for positive constants $k_1$, $k_2$, and $k_3$, then the equilibrium point at the origin is exponentially stable.

4. **Perko, Section 2.12, problem 2** Use Theorem 1 [Center Manifold Theorem] to determine the qualitative behaviour near the non-hyperbolic critical point at the origin for the system

$$
\begin{align*}
x & = y \\
\dot{y} & = -y + \alpha x^2 + xy
\end{align*}
$$

for $\alpha \neq 0$ and for $\alpha = 0$; i.e., follow the procedure in Example 1 after diagonalizing the system as in Example 3.
5. Consider the following system in $\mathbb{R}^2$:

\[
\begin{align*}
\dot{x} &= -\frac{\alpha}{2}(x^2 + y^2) + \alpha(x + y) - \alpha \\
\dot{y} &= -\alpha xy + \alpha(x + y) - \alpha
\end{align*}
\]

Determine the stable, unstable, and centre manifold of the equilibrium point at $(x, y) = (1, 1)$, and determine the stability of this equilibrium point for $\alpha \neq 0$. For determining stability, note that near the equilibrium point there are two 1-dimensional invariant linear manifolds of the form $M = \{(a_1, a_2) \in \mathbb{R}^2 \mid a_2 = k a_1\}$; determine the flow on these invariant manifolds.


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