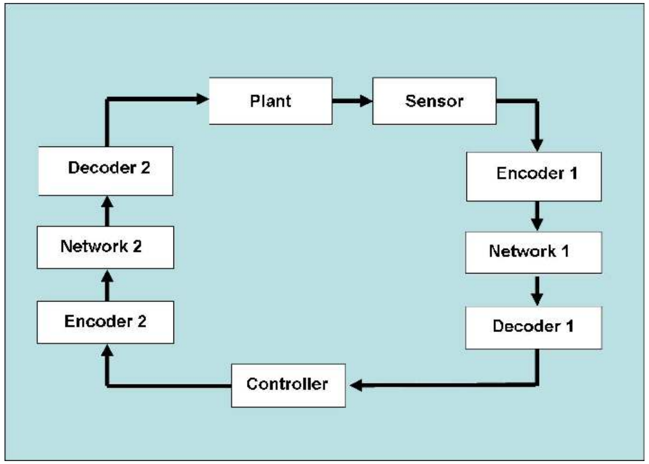
CDS 270-2: Lecture 6-1
Towards a Packet-based Control Theory

Ling Shi

May 1 2006

- **Goals:**
  - Describe main issues with a packet-based control system
  - Introduce common models for a packet-based control system
  - Learn basic techniques to solve the issues
- **Reading:**
  - Ling Shi, Michael Epstein and Richard Murray, "*Networked Control Systems with Norm Bounded Uncertainties: A Stability Analysis*", to appear in ACC 06
  - Ling Shi, Michael Epstein and Richard Murray, "*Towards Robust Control Over a Packet-based Network* ", to appear in MTNS 06

Issues and Models of NCS



Issues

- Packet Drops
  - network congestion
  - poor link quality (Wed's lectures)
- Finite Data Rate
  - Physical constraints
- Network Induced Delays
  - Token Ring network
  - Ethernet

Models

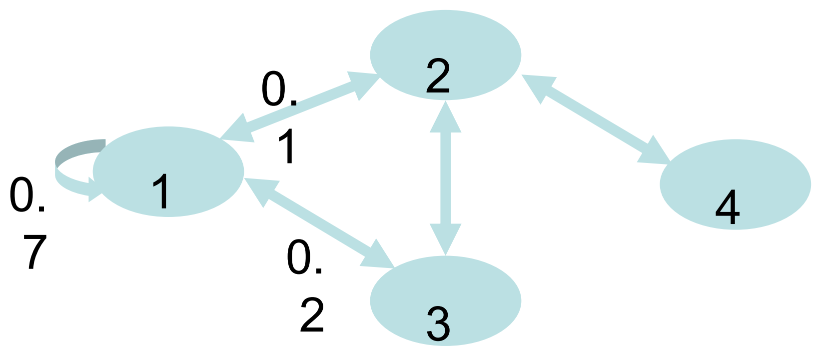
- Packet Drops
  - i.i.d drops
  - hidden Markov Chain
- Finite Data Rate
  - finite bits per time
  - coding & decoding
  - quantization
- Network Induced Delays
  - fixed delays
  - bounded delays
  - random delays

Some Background

- Consider the Jump Linear System (JLS)

$$x_{k+1} = A(\sigma_k)x_k \ , \ \sigma_k \in \{1, 2, \cdots, N\}$$

with steady state probability distribution $\mathbf{Pr}[\sigma_k = j] = \pi_j$



A Markov Chain with 4 States

$$P = [P_{ij}] = \begin{bmatrix} 0.7 & 0.1 & 0.2 & 0 \\ 0.1 & 0.5 & 0.2 & 0.2 \\ 0.4 & 0.3 & 0.3 & 0 \\ 0 & 0.1 & 0 & 0.9 \end{bmatrix}$$

$$\pi_j = \pi_j P$$

Some Background

- Any system of the form above with inherent randomness ω is called almost surely stable if

$$\mathbf{Pr}[\lim_{k\to\infty} |x_k(x_0,\omega)| = 0] = 1$$

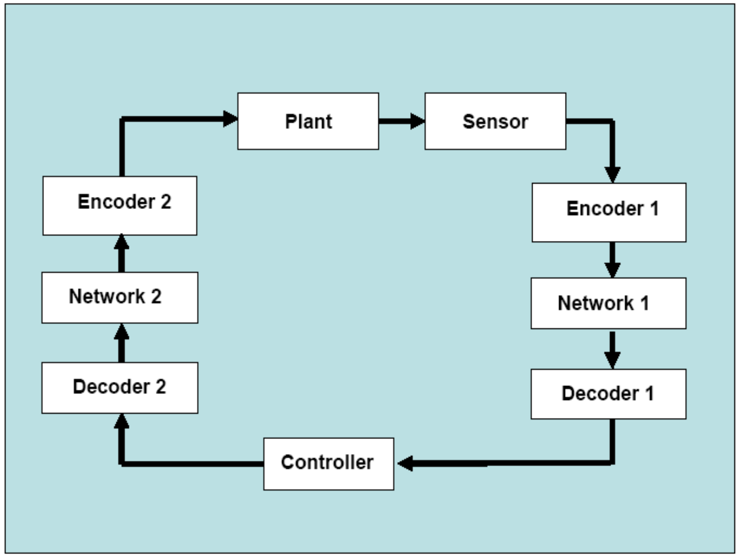
- Comparison with Lyapunov stability.

- Lemma 1: The JLS above is almost sure stable if there exists a matrix norm $||\cdot||$ such that

$$\prod_{i=1}^{N} ||A(i)||^{\pi_i} < 1$$

Proof: (intuitive ideas)

ACC 06: Packet Drops and Fixed Data Rate



NCS Type II

$$\begin{aligned} x_{k+1} &= (A + \Delta_k) x_k + \gamma_k B \bar{u}_k, \\ y_k &= \lambda_k C x_k, \\ u_k &= u_k(\bar{y}_k) \end{aligned}$$

Assumptions:

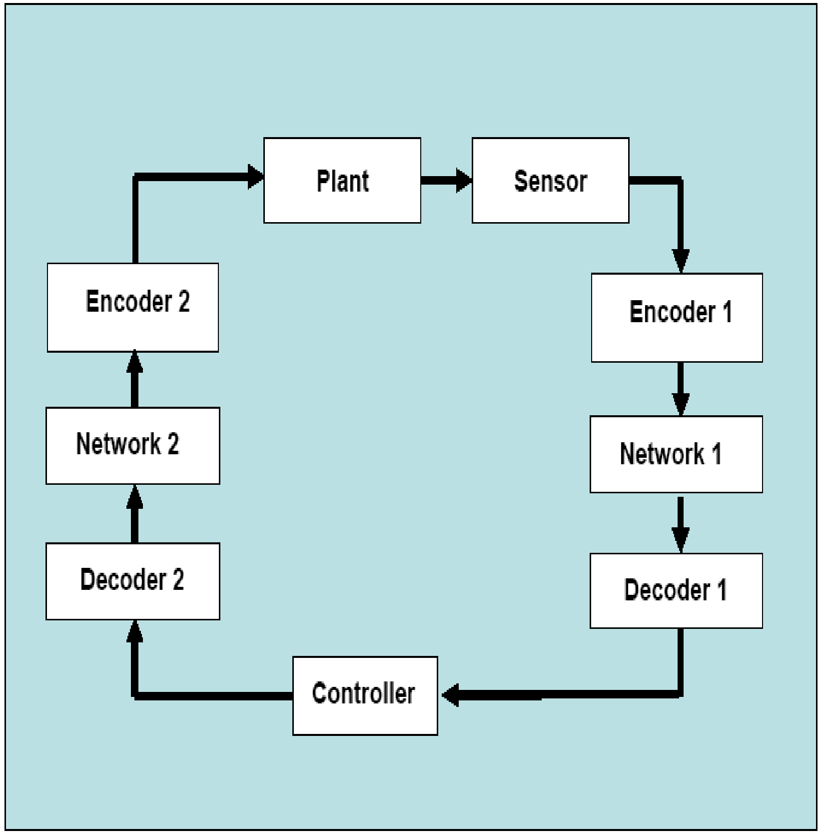
- γ_k, λ_k are i.i.d Bernoulli RVs and have mean γ and λ
- $\Delta_k^T \Delta_k \leq K^2 I$
- Network 1 has data rate R_1
- Network 2 has data rate R_2
- Everything is synchronized
- No packet reordering or delays

Notations:

- $|A|$: Induced matrix norm
- log is of base 2
- Almost sure stability

What are the conditions on the network and system parameters such that closed loop stability is guaranteed?

Stability Analysis for NCS II

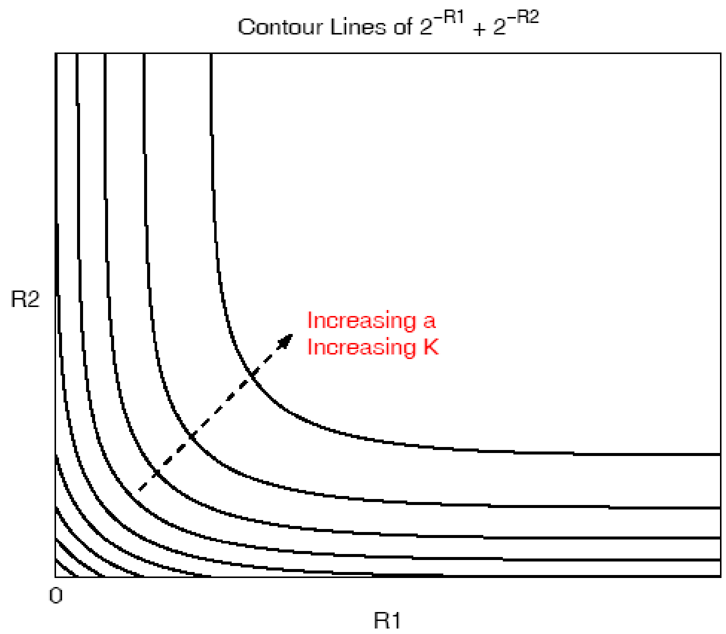


NCS Type II

- Lemma 1: $\lambda = 1$, $\gamma = 1$ and $K = 0$ imply closed loop stability if

$$a(2^{-R_1} + 2^{-R_2}) < 1$$

Proof:



Contour plot of $2^{-R1}+2^{-R2}$. The lines depict fixed values of a and K. Regions above the lines are stable for these fixed values.

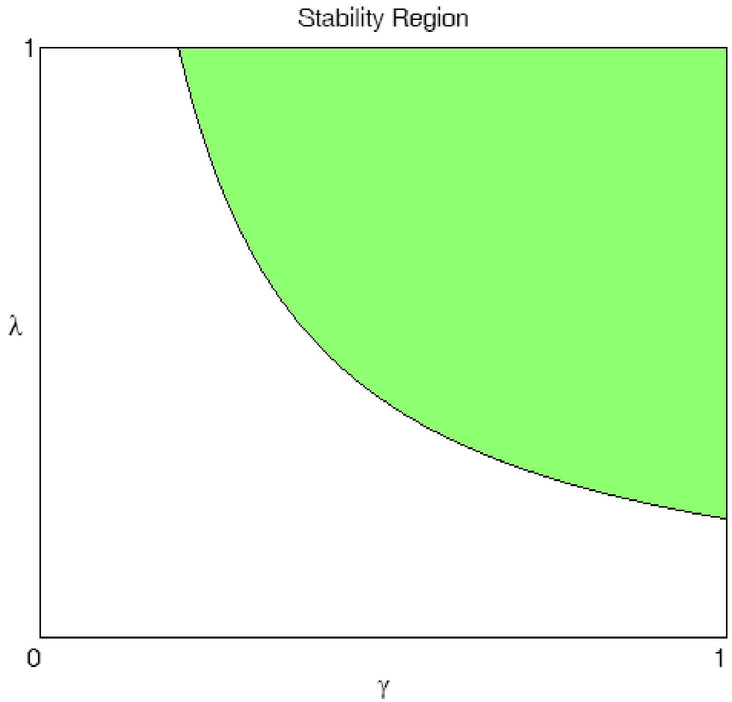
Stability Analysis for NCS II

- Lemma 2: $\lambda = 1$, $\gamma = 1$ and $K < 1$ imply closed loop stability if

$$a(2^{-R_1} + 2^{-R_2}) < 1 - K$$

- Lemma 3: $R_1 = R_2 = \infty$, $K < 1$ imply closed loop stability if

$$\lambda\gamma > \frac{\log(a+K)}{\log(a+K) - \log K}$$



Stability plot for NCS II with packet arrival rates γ and λ

- Lemma 4: A sufficient condition for closed loop stability is that

$$(a+K)^{1-\lambda\gamma}(a2^{-R_1} + a2^{-R_2} + K)^{\lambda\gamma} < 1$$

Stability Analysis for NCS II

• Theorem 2: Assume *B*, *C* are invertible and the system dimension is *n*. Then a sufficient condition for closed loop stability is that

$$(|A|+K)^{1-\lambda\gamma}(|A|2^{-\frac{R_1}{n}}+|B||B^{-1}A|2^{-\frac{R_2}{n}}+K)^{\lambda\gamma} < 1$$

• Remark: All previous lemmas and theorems can be derived from this master inequality including NCS I.

Simulation Examples

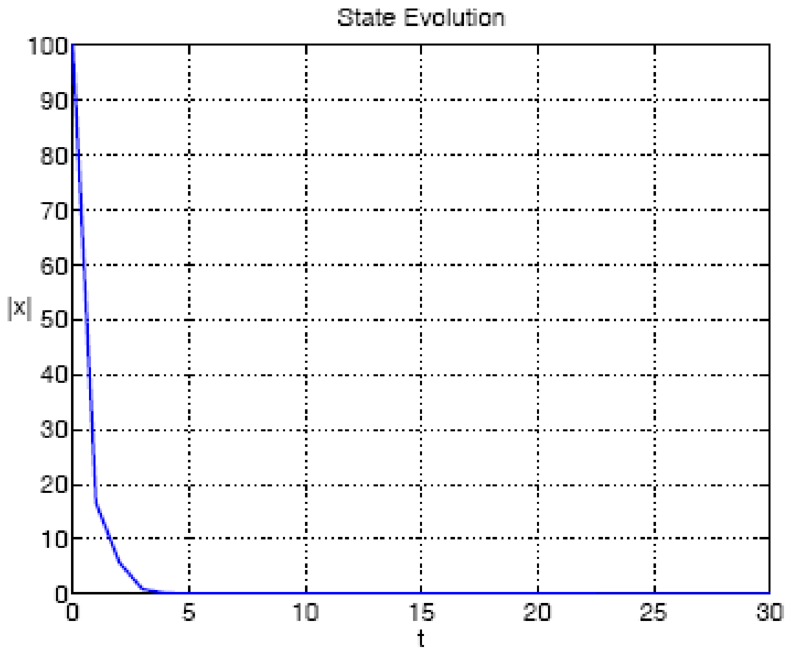


Fig. 6. System in Example 14 with $R_1 = R_2 = 5, \lambda = \gamma = 1$.

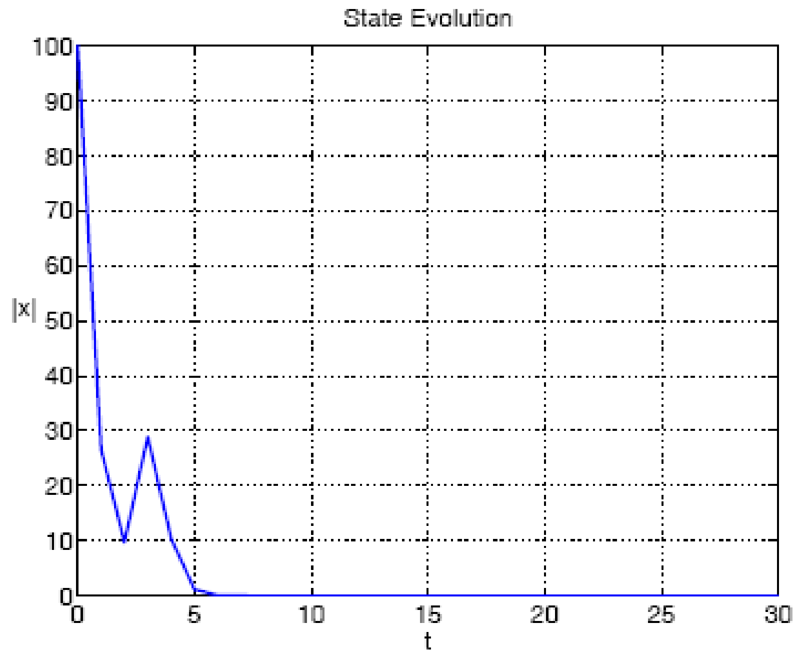


Fig. 7. System in Example 14 with $R_1 = R_2 = \infty, \lambda = 0.85, \gamma = 0.8$.

Simulation Examples

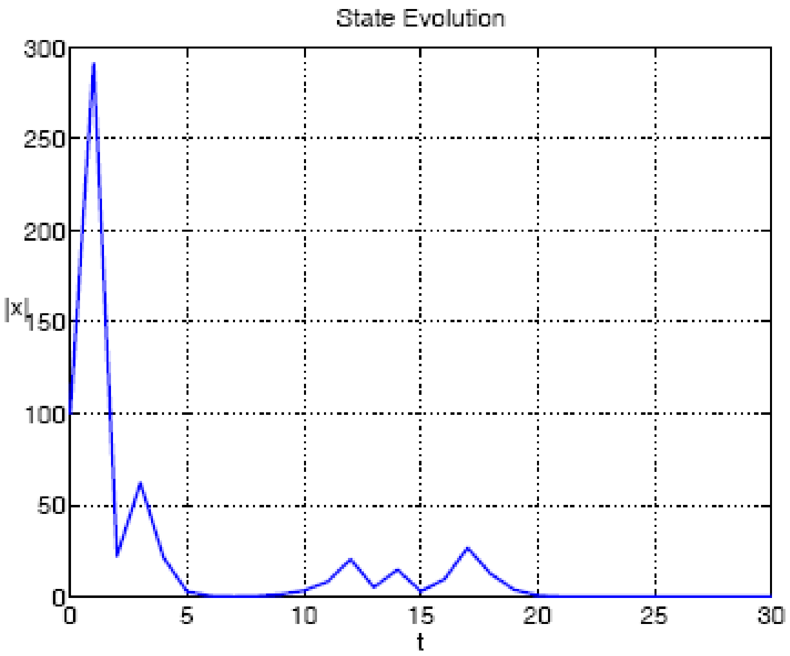


Fig. 8. System in Example 14 with $R_1 = R_2 = 5, \lambda = 0.85, \gamma = 0.8$.

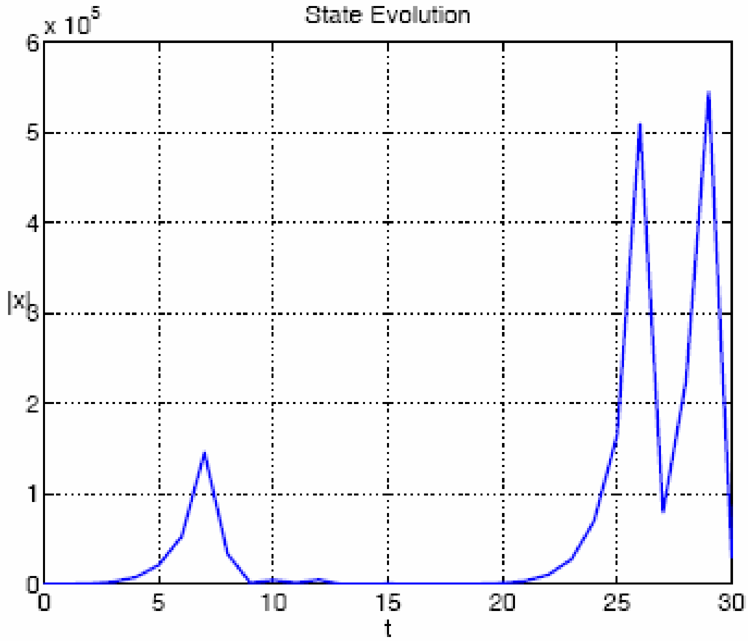


Fig. 9. System in Example 14 with $R_1 = 5, R_2 = 3, \lambda = 0.5, \gamma = 0.6$.

Future work

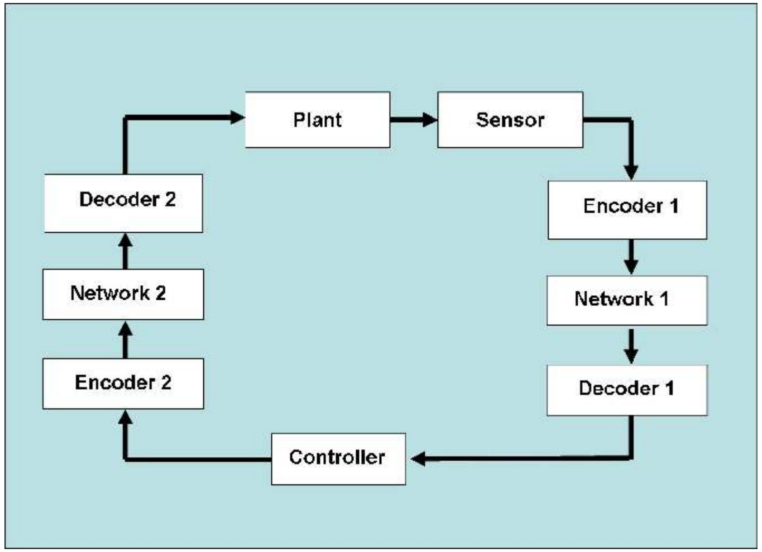
• What if *B*, *C* are not invertible ? (solved in MTNS 06)

• Necessary conditions and how far are the sufficient conditions away from them?

• Uncertainties in *B* and *C*

• Other types of uncertainty models

MTNS06: Packet Drops Only



$$
\begin{aligned}
x_{k+1} &= (A + \Delta_k)x_k + \gamma_k B u_k, \\
y_k &= \lambda_k C x_k
\end{aligned}
$$

- (*A*,*B*) Controllable, (*C*,*A*) Observable
  - Output feedback
  - Design observer
- Assume infinite bandwidth & no delays
- $||\Delta_k|| \leq \beta$
- γ_k, λ_k are i.i.d Bernoulli RVs

Analyze 4 options for this NCS

- Zero Control, Observer at the Sensor
- Zero Control, Observer at the Controller
- Anticipatory Control, Observer at the Sensor
- Anticipatory Control, Observer at the Controller

Some Background

- Lemma 2: If *A* is stable, then there exists a matrix norm $||\cdot||_H$ such that $||A||_H < 1$ where

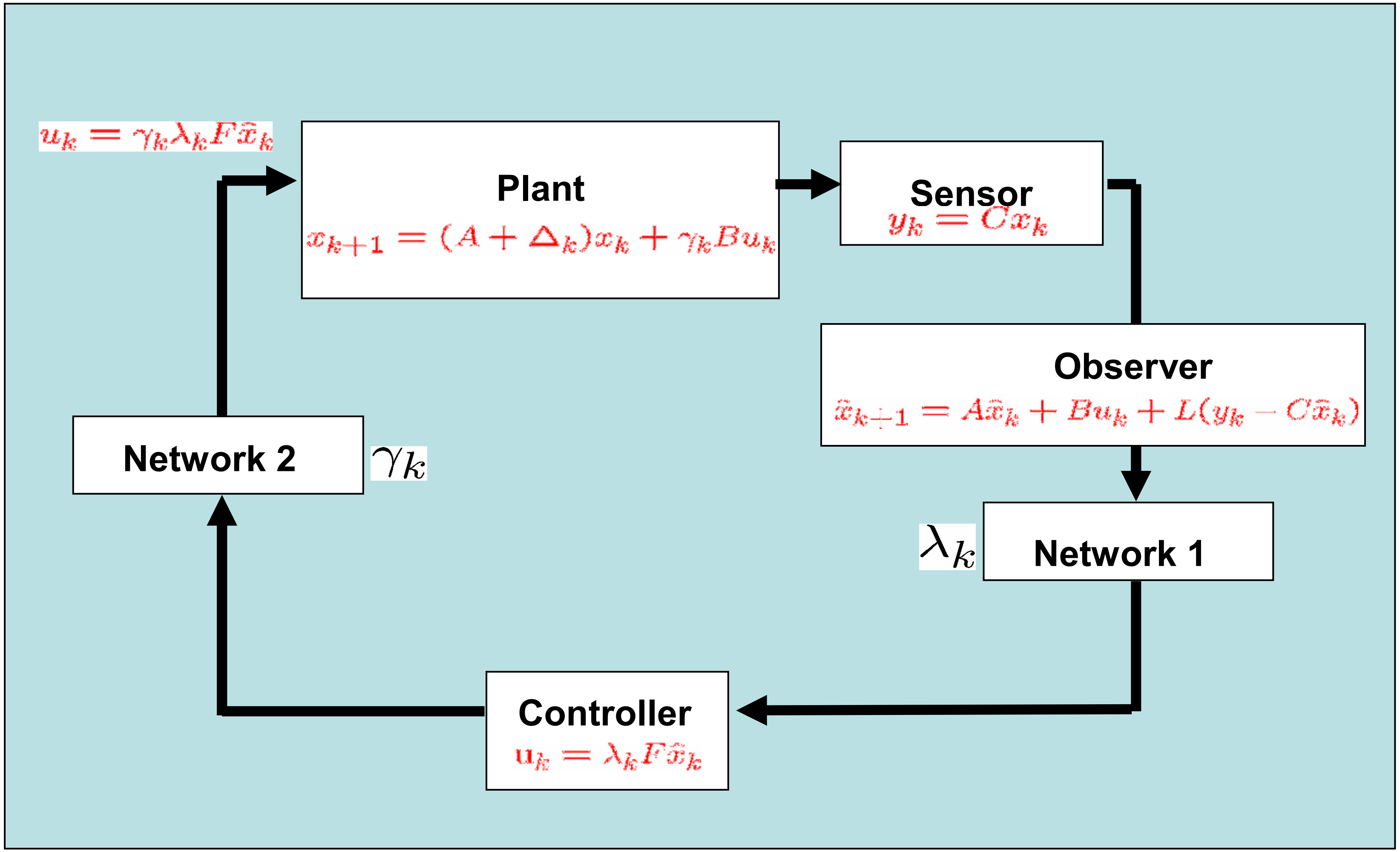
$$||A||_H^2 = \sup_{x \neq 0} \frac{||Ax||_H^2}{||x||_H^2} = \sup_{x \neq 0} \frac{x^T A^T H A x}{x^T H x}$$

  Proof:

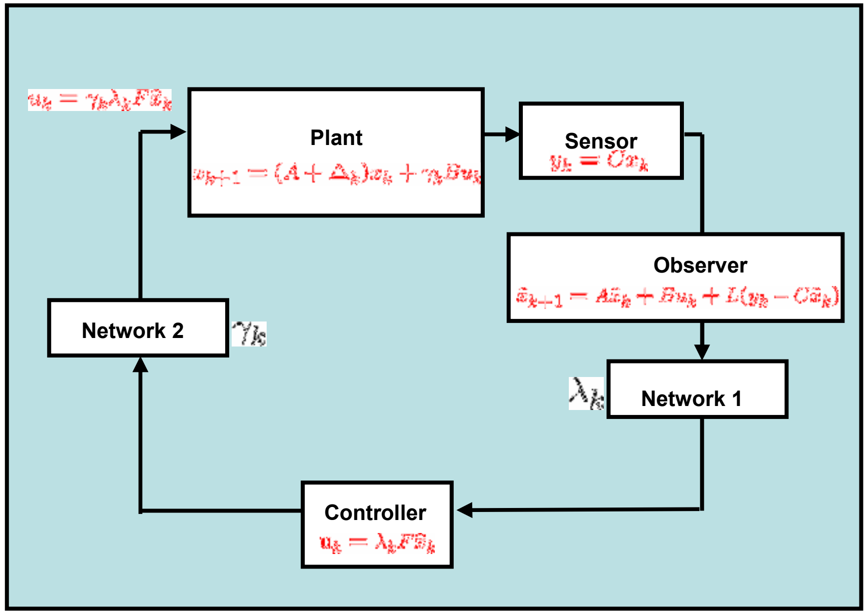
- Corollary 2: For any *A*, the following identity is true

$$||A||_H = ||PAP^{-1}||, \ P = H^{1/2}.$$

Zero Control, Observer at Sensor



Zero Control, Observer at Sensor



$$\begin{bmatrix} x_{k+1} \\ e_{k+1} \end{bmatrix} = A(\sigma_k) \begin{bmatrix} x_k \\ e_k \end{bmatrix}$$

$$A(\sigma_k) = \begin{cases} M_1 + T_k, & \text{if } \lambda_k \gamma_k = 1 \\ M_2 + T_k, & \text{if } \lambda_k \gamma_k = 0 \end{cases}$$

with

$$M_1 = \begin{bmatrix} A + BF & -BF \\ 0 & A - LC \end{bmatrix}$$
$$M_2 = \begin{bmatrix} A & 0 \\ 0 & A - LC \end{bmatrix}$$
$$T_k = \begin{bmatrix} \Delta_k & 0 \\ \Delta_k & 0 \end{bmatrix}.$$

Let $\beta < \beta_{max} = \frac{1 - ||M_1||_H}{\sqrt{2}||P|| \cdot ||P^{-1}||}$

then the NCS is almost sure stable if

$$N_1^{\lambda\gamma} N_2^{1-\lambda\gamma} < 1$$

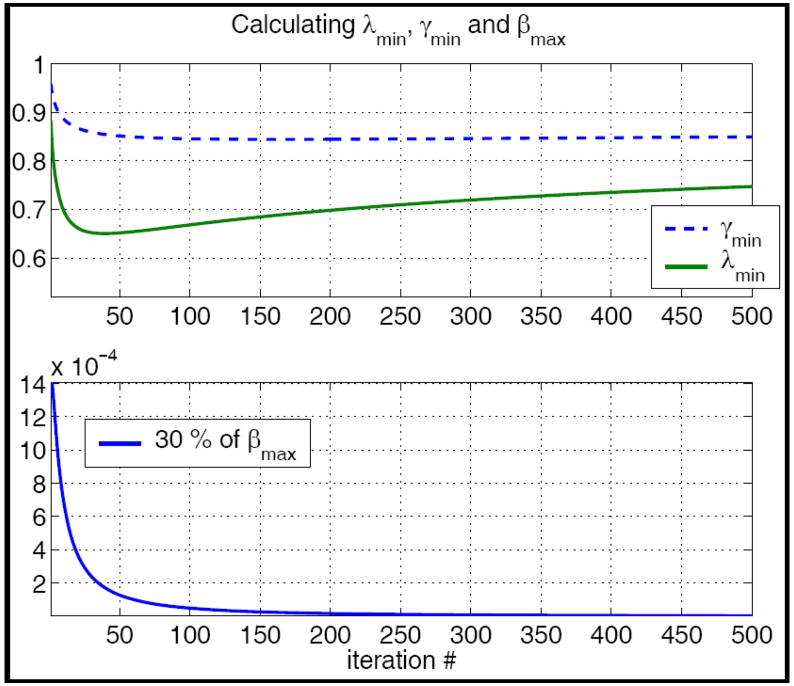
Proof:

Design Tradeoffs

- Objective: maximize β, minimize λ and γ according to

$$\beta_{max} = \frac{1 - ||M_1||_H}{\sqrt{2}\,||P|| \cdot ||P^{-1}||}$$

$$N_1^{\lambda\gamma} N_2^{1-\lambda\gamma} < 1$$



Algorithm II

Given (*A, B, C, F, L*)

1. Form the matrices M_j as in Section III, where $j \in \{1,2\}$ or $\{1,2,3,4\}$ depending on different control schemes.
2. Set i = 1 and $Q_i = I$
3. Solve $A^T H_i A - H_i = -Q_i$ via standard lyapunov equation solvers to get H_i. Set $Q_i = H_i$.
4. Decompose H_i into $H_i = P_i P_i$ via standard algorithms.
5. Find maximum sufficient uncertainty according to (12) or (36) depending on different control schemes.
6. Find minimum sufficient packet arrival rates λ and γ for some portion of the uncertainty found in step 5 according to Theorems 5,7,8, or 9 respectively.
7. $i = i+1$.
8. Repeat steps 3 to 7 until the incremental increase or decrease of these values are within a certain threshold.

Future Work

- Necessary conditions and how far are the sufficient conditions away from them?

- Other types of uncertainty models

- UDP estimation? (partially solved in CDC 06)