Perko Section 2.1

4. Show that the initial value problem

$$\dot{x} = x^3$$
$$x(0) = 2$$

has a solution on an interval $(-\infty, b)$ for some $b \in \mathbf{R}$. Sketch the solution in the (t, x)-plane and note the behavior of $x(t)$ as $t \to b^-$.

Perko Section 2.2

3. Use the method of successive approximations to show that if $\mathbf{f}(\mathbf{x}, t)$ is continuous in t for all t in some interval containing $t = 0$ and continuously differentiable in $\mathbf{x}$ for all $\mathbf{x}$ in some open set $E \subset \mathbf{R}^n$ containing $\mathbf{x}_0$, then there exists an $a > 0$ such that the initial value problem

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$
$$\mathbf{x}(0) = \mathbf{x}_0$$

has a unique solution $\mathbf{x}(t)$ on the interval $[-a, a]$. **Hint:** Define $\mathbf{u}_0(t) = \mathbf{x}_0$ and

$$\mathbf{u}_{k+1}(t) = \mathbf{x}_0 + \int_0^t \mathbf{f}(\mathbf{u}_k(s), s)\, ds$$

and show that the successive approximations $\mathbf{u}_k(t)$ converge uniformly to $\mathbf{x}(t)$ on $[-a, a]$ as in the proof of the fundamental existence-uniqueness theorem.

Perko Section 2.3

2. (a) Solve the initial value problem

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$
$$\mathbf{x}(0) = \mathbf{y}$$

for $\mathbf{f}(\mathbf{x}) = (-x_1, -x_2 + x_1^2, x_3 + x_1^2)^T$. Denote the solution by $\mathbf{u}(t, \mathbf{y})$ and compute

$$\Phi(t, \mathbf{y}) = \frac{\partial \mathbf{u}}{\partial \mathbf{y}}(t, \mathbf{y}).$$

Compute the derivative $D\mathbf{f}(\mathbf{x})$ for the given function $\mathbf{f}(\mathbf{x})$ and show that for all $t \in \mathbf{R}$ and $\mathbf{y} \in \mathbf{R}^3$, $\Phi(t, \mathbf{y})$ satisfies

$$\dot{\Phi} = A(t, \mathbf{y})\Phi$$
$$\Phi(0, \mathbf{y}) = I$$

where $A(t, \mathbf{y}) = D\mathbf{f}[\mathbf{u}(t, \mathbf{y})]$.

Perko Section 2.4

2. Find the maximal interval of existence (α, β) for the following initial value problems and if $\alpha > -\infty$ or $\beta < \infty$ discuss the limit of the solution as $t \to \alpha^+$ or as $t \to \beta^-$ respectively:

(a) $$\begin{aligned} \dot{x}_1 &= x_1^2 & x_1(0) &= 1 \\ \dot{x}_2 &= x_2 + x_1^{-1} & x_2(0) &= 1 \end{aligned}$$

(b) $$\begin{aligned} \dot{x}_1 &= \frac{1}{2x_1} & x_1(0) &= 1 \\ \dot{x}_2 &= x_2^2 & x_2(0) &= 1 \end{aligned}$$

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{1}$$

$$\dot{\mathbf{x}} = A\mathbf{x}, \tag{2}$$

Perko Section 2.5

5. Determine the flow ϕ_t: $\mathbf{R}^2 \to \mathbf{R}^2$ for the nonlinear system (1) with

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} -x_1 \\ 2x_2 + x_1^2 \end{bmatrix}$$

and show that the set $S = \{\mathbf{x} \in \mathbf{R}^2 \mid x_2 = -x_1^2/4\}$ is invariant with respect to the flow ϕ_t.

Perko Section 2.6

3. Show that the continuous map H: $\mathbf{R}^3 \to \mathbf{R}^3$ defined by

$$H(\mathbf{x}) = \begin{bmatrix} x_1 \\ x_2 + x_1^2 \\ x_3 + \frac{x_1^2}{3} \end{bmatrix}$$

has a continuous inverse H^{-1}: $\mathbf{R}^3 \to \mathbf{R}^3$ and that the nonlinear system (1) with

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} -x_1 \\ -x_2 + x_1^2 \\ x_3 + x_1^2 \end{bmatrix}$$

is transformed into the linear system (2) with $A = D\mathbf{f}(\mathbf{0})$ under this map; i.e., if $\mathbf{y} = H(\mathbf{x})$, show that $\dot{\mathbf{y}} = A\mathbf{y}$.