CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 101/110

Homework Set #3

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Due: 16 Oct 06

Note: In the upper left hand corner on the back of the first page of your homework set, please put the number of hours that you spent on this homework set (including reading).

All students should complete the following problems:

1. For each of the following systems, locate the equilibrium points for the system and indicate whether each is asymptotically stable, stable (but not asymptotically stable), or unstable. To determine stability, you can either use a phase portrait (if appropriate) or simulate the system using multiple nearby initial conditions to how the state evolves. (Note: if you know how to check stability through the linearization, you can also use this approach.)

   (a) *Nonlinear spring mass.* Consider a nonlinear spring mass system,

   $$m\ddot{x} = -k(x - ax^3) - b\dot{x},$$

   where $m = 1000$ kg is the mass, $k = 250$ kg/sec^2 is the nominal spring constant, $a = 0.01$ represents the nonlinear "softening" of the spring, and $b = 100$ kg/sec is the damping coefficient. Note that this is very similar to the spring mass system we have studied in class, except for the nonlinearity.

   (b) *Genetic toggle switch.* A genetic toggle switch can be built with two proteins u and v that regulate production of each other in a symmetric fashion. This system exhibits two behaviors, one where expression of u is high while v is low and a second behavior where v is high while u is low. The genetic toggle switch mechanism can be modeled as

   $$\begin{aligned} \dot{u} &= \frac{10}{1+v^2} - u \\ \dot{v} &= \frac{10}{1+u^2} - v, \end{aligned}$$

   where u and v are protein concentrations. The nonlinear term in the equations is a model for repression, where v represses production of u and visa versa. The linear term describes degradation of the protein.

   (c) *Congestion control of the Internet.* A simple model for congestion control between N computers connected by a router is given by the differential equation

   $$\dot{x}_i = -b\frac{x_i^2}{2} + (b_{\max} - b)$$
   $$\dot{b} = \left(\sum_{i=1}^{N} x_i\right) - c$$

where $x_i \in \mathbb{R}$, $i = 1, N$ are the transmission rates for the sources of data, $b \in \mathbb{R}$ is the current buffer size of the router, $b_{\text{max}} > 0$ is the maximum buffer size, and $c > 0$ is the capacity of the link connecting the router to the computers. The $\dot{x}_i$ equation represents the control law that the individual computers use to determine how fast to send data across the network (this version is motivated by a protocol called "Reno") and the $\dot{b}$ equation represents the rate at which the buffer on the router fills up. Consider the case where $N = 2$ (so that we have three states, x_1, x_2, and b), and take $b_{\text{max}} = 1$ Mb and $c = 2$ Mb/sec.

2. Consider the cruise control system described in Section 3.1 of Åström and Murray. Generate a phase portrait for the closed loop system on flat ground ($\theta = 0$), in third gear, using a PI controller (with $k_p = 0.5$ and $k_i = 0.1$), $m = 1000$ kg and desired speed 20 m/s. Your system model should include the effects of saturating the input between 0 and 1.

   Keep in mind that when modeling feedback control, additional states can arise that do not appear in the original dynamics.

Only CDS 110a students need to complete the following additional problems:

3. Find a Lyapunov function for the cruise control system in the previous problems, showing that the system is locally asymptotically stable at the desired speed. If you like, you can use the specific parameters listed above, although it is also possible to solve the problem leaving parameter values unspecified (with some assumptions, which you should state).

   We will be doing an example in class on Friday that will introduce techniques that will be helpful for this problem.

4. (a) Show that if v is an eigenvector of a matrix A, corresponding to eigenvalue λ, then v is also an eigenvector of the matrix $\exp(A)$. What is the corresponding eigenvalue?

   (b) Use analytic methods from the lecture notes from the linear algebra review to compute $\exp(A)$, where

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

   You should do this computation by hand (not using Mathematica or Maple).