

# Lecture 5: Control Over Lossy Networks

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## 1 Classical LQG Control

The system:

$$\begin{aligned}x_{k+1} &= Ax_k + Bu_k + w_k, \\y_k &= Cx_k + v_k\end{aligned}$$

$x_0 \sim \mathcal{N}(0, \Sigma)$ ,  $w_k \sim \mathcal{N}(0, Q)$ ,  $v_k \sim \mathcal{N}(0, R)$ .

Information available for the controller at time  $k$ :

$$Y_k = (y_0, \dots, y_k).$$

The control at time  $k$  is a function of the information  $Y_k$ :  $u_k(Y_k)$ .

The goal of a finite horizon LQG problem is to find a controller that minimizes the following quadratic cost:

$$J(N) = \min_{u_0, \dots, u_N} \mathbb{E} \sum_{k=0}^N (x_k^T W x_k + u_k^T U u_k).$$

### 1.1 Optimal Estimator Design

Since the system is linear, the following Kalman filtering equations holds:

1. **Initialization:**

$$\hat{x}_{0|-1} = 0, P_{0|-1} = \Sigma. \quad (1)$$

2. **Prediction:**

$$\hat{x}_{k+1|k} = A\hat{x}_k + Bu_k, P_{k+1|k} = AP_kA^T + Q. \quad (2)$$

3. **Correction:**

$$\hat{x}_{k+1} = \hat{x}_{k+1|k} + P_{k+1|k}C^T(CP_{k+1|k}C^T + R)^{-1}(y_{k+1} - C\hat{x}_{k+1|k}), \quad (3)$$

$$P_{k+1} = P_{k+1|k} - P_{k+1|k}C^T(CP_{k+1|k}C^T + R)^{-1}CP_{k+1|k}. \quad (4)$$

And we have that

$$\mathbb{E}(x_k|Y_k) = \hat{x}_k, \text{Cov}(x_k|Y_k) = P_k.$$

One important thing to notice: the  $P_k$  is independent from  $u_k$ . This is because the system is linear and hence we can subtract  $u_k$ .

## 1.2 Optimal Controller Design and Separation Principle

Now we can try to solve the optimal control design problem by dynamical programming.

Define the value function

$$V(t) = \min_{u_t, \dots, u_N} \mathbb{E} \left[ \sum_{k=t}^N (x_k^T W x_k + u_k^T U u_k) \right]$$

Clearly

$$J(N) = V(0),$$

and

$$V(N) = \mathbb{E} x_N^T W x_N.$$

Now, by Bellman equation

$$V(t) = \min_{u_t} \mathbb{E} (x_t^T W x_t + u_t^T U u_t + V(t+1)) \quad (5)$$

We will guess that

$$V(t) = \mathbb{E} x_t^T S_t x_t + c_t \quad (6)$$

To prove this, we will use induction.

Clearly  $S_N = W$  and  $c_N = 0$ .

Now suppose (6) holds for  $t+1$ , and we look at the following quantity:

$$\begin{aligned} & \mathbb{E} (x_t^T W x_t + u_t^T U u_t + V(t+1)) \\ &= \mathbb{E} x^T (W + A^T S_{t+1} A) x + \text{tr}(S_{t+1} Q) + c_{t+1} \\ &+ \mathbb{E} [u_t^T (U + B^T S_{t+1} B) u_t + x_t^T A^T S_{t+1} B u_t + u_t^T B^T S_{t+1} A x_t] \end{aligned}$$

Notice that the controller do not know  $x_t$ . Hence, let us rewrite  $x_t$  as

$$x_t = \hat{x}_t + x_t - \hat{x}_t = \hat{x}_t + e_t.$$

**Theorem 1.** 1.  $e_k$  is independent of  $Y_k$  and hence  $\hat{x}_k$  and  $u_k$ .

2.

$$\mathbb{E} \hat{x}_k^T S \hat{x}_k = \mathbb{E} x_k^T S x_k - \text{tr}(S P_k)$$

*Proof.* 1. Notice that

$$\mathbb{E}(e_k | Y_k) = \mathbb{E}(x_k | Y_k) - \mathbb{E}(\hat{x}_k | Y_k) = \hat{x}_k - \hat{x}_k = 0.$$

Hence,  $e_k$  is **linearly** independent from  $Y_k$ . Since  $e_k$  and  $Y_k$  are jointly Gaussian,  $e_k$  and  $Y_k$  are independent.

2. Since

$$\mathbb{E} x_k^T S x_k = \mathbb{E} \hat{x}_k^T S \hat{x}_k + \mathbb{E} \hat{x}_k^T S e_k + \mathbb{E} e_k^T S \hat{x}_k + \mathbb{E} e_k^T S e_k = \mathbb{E} \hat{x}_k^T S \hat{x}_k + 0 + 0 + \text{tr}(S P_k)$$

□

Now let us look at

$$\begin{aligned}
& \mathbb{E} [u_t^T (U + B^T S_{t+1} B) u_t + x_t^T A^T S_{t+1} B u_t + u_t^T B^T S_{t+1} A x_t] \\
&= \mathbb{E} [u_t^T (U + B^T S_{t+1} B) u_t + \hat{x}_t^T A^T S_{t+1} B u_t + u_t^T B^T S_{t+1} A \hat{x}_t] \\
&= \mathbb{E} [(u_t - u_t^*)^T (U + B^T S_{t+1} B) (u_t - u_t^*) - \hat{x}_t^T A^T S_{t+1} B (U + B^T S_{t+1} B)^{-1} B^T S_{t+1} A \hat{x}_t]
\end{aligned}$$

where  $u_t^* = -(U + B^T S_{t+1} B)^{-1} B^T S_{t+1} A \hat{x}_t$ . Hence

$$\begin{aligned}
V(t) &= \mathbb{E} x_t^T (W + A^T S_{t+1} A - A^T S_{t+1} B (U + B^T S_{t+1} B)^{-1} B^T S_{t+1} A) x_t \\
&\quad + c_{t+1} + \text{tr}(A^T S_{t+1} B (U + B^T S_{t+1} B)^{-1} B^T S_{t+1} A P_k) + \text{tr}(S_{t+1} Q)
\end{aligned}$$

Therefore

$$S_t = W + A^T S_{t+1} A - A^T S_{t+1} B (U + B^T S_{t+1} B)^{-1} B^T S_{t+1} A, \quad (7)$$

and

$$c_t = c_{t+1} + \text{tr}(A^T S_{t+1} B (U + B^T S_{t+1} B)^{-1} B^T S_{t+1} A P_k) + \text{tr}(S_{t+1} Q).$$

Thus,

$$J(N) = \mathbb{E}(x_0^T S_0 x_0) + c_0 = \text{tr}(S_0 \Sigma) + c_0.$$

### 1.3 Infinite Horizon LQG problem

Define  $J$  as

$$J = \lim_{N \rightarrow \infty} \frac{J(N)}{N}.$$

We consider the problem of finding a controller that minimizes the infinite horizon cost  $J$ .

Notice that (7) is a Riccati equation. Hence, if  $N \rightarrow \infty$ , then  $S_k$  converges to  $S$ , which is the fixed solution of

$$S = W + A^T S A - A^T S B (U + B^T S B)^{-1} B^T S A, \quad (8)$$

The optimal controller is given by

$$u_k^* = -(U + B^T S B)^{-1} B^T S A \hat{x}_k.$$

## 2 Witsenhausen's Counterexample

Consider  $x_0 \sim \mathcal{N}(0, \sigma^2)$ .

1. The first player knows  $x_0$  and he computes an

$$x_1 = f(x_0),$$

which is a function of  $x_0$ .

- The first player sends  $x_1$  to the second player through a noisy channel. Therefore, the second player receives

$$y_2 = x_1 + v,$$

where  $v \sim \mathcal{N}(0, 1)$ .

- The second player then computes  $x_2 = g(y_2)$ .

The goal is to minimize the following cost function

$$J = \min_{f,g} \mathbb{E} k^2(x_0 - x_1)^2 + (x_1 - x_2)^2$$

Alternatively, one can consider the following equivalent scheme:

- The controller knows  $x_0$  and it computes an control  $u$

$$u = f(x_0),$$

which is a function of  $x_0$ .

- The state of the system satisfies the following update equation:

$$x_1 = x_0 + u.$$

- The second player observe the system via a noisy sensor:

$$y_2 = x_1 + v,$$

where  $v \sim \mathcal{N}(0, 1)$ .

- The second player then computes the state estimate  $x_2 = g(y_2)$ .

The goal is to minimize the following cost function

$$J = \min_{f,g} \mathbb{E} k^2 u^2 + (x_1 - x_2)^2$$

## 2.1 Optimal linear strategy

We adopt the first setting. Consider that both  $f(x) = \lambda x$  and  $g(x) = \mu x$  are linear, then

$$J = \min_{\lambda, \mu} \mathbb{E} k^2 (1 - \lambda)^2 x_0^2 + (\lambda x_0 - \mu(\lambda x_0 + v))^2$$

Therefore,

$$\mu = \frac{\lambda^2 \sigma^2}{1 + \lambda^2 \sigma^2},$$

and

$$\lambda = \arg \min_{\lambda} k^2 \sigma^2 (1 - \lambda)^2 + \frac{\lambda^2 \sigma^2}{1 + \lambda^2 \sigma^2}.$$

If  $k^2 \sigma^2 = 1$  and  $k \rightarrow 0$ , then  $\lambda \approx 1$  and  $J \approx 1$ .

### 3 Nonlinear strategy

One can prove that for small  $k$  and  $k^2\sigma^2 = 0$ , the following design is better than the linear design:

$$f(x) = \sigma \operatorname{sgn}(x), \quad g(x) = \sigma \frac{1 - e^{-2\sigma x}}{1 + e^{-2\sigma x}}.$$

### 4 Control Over Lossy Networks

The system:

$$\begin{aligned} x_{k+1} &= Ax_k + \nu_k Bu_k + w_k, \\ y_k &= Cx_k + v_k \end{aligned}$$

$x_0 \sim \mathcal{N}(0, \Sigma)$ ,  $w_k \sim \mathcal{N}(0, Q)$ ,  $v_k \sim \mathcal{N}(0, R)$ .

$\nu_k$  is an i.i.d. Bernouli process with  $P(\nu_k = 1) = \lambda$ .

The goal of a finite horizon LQG problem is to find a controller that minimizes the following quadratic cost:

$$J = \min_{u_0, \dots, u_N} \lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \sum_{k=0}^N (x_k^T W x_k + \nu_k u_k^T U u_k).$$

#### 4.1 TCP case

Information available for the controller at time  $k$ :

$$\mathcal{I}_k = (y_0, \dots, y_k, \nu_0, \dots, \nu_k).$$

$J$  is finite if and only if the following Riccati equation has a positive semidefinite solution:

$$S = W + A^T S A - \lambda A^T S B (U + B^T S B)^{-1} B^T S A, \quad (9)$$

Optimal Filter:

**1. Initialization:**

$$\hat{x}_{0|-1} = 0, \quad P_{0|-1} = \Sigma. \quad (10)$$

**2. Prediction:**

$$\hat{x}_{k+1|k} = A\hat{x}_k + \nu_k Bu_k, \quad P_{k+1|k} = AP_k A^T + Q. \quad (11)$$

**3. Correction:**

$$\hat{x}_{k+1} = \hat{x}_{k+1|k} + P_{k+1|k} C^T (C P_{k+1|k} C^T + R)^{-1} (y_{k+1} - C \hat{x}_{k+1|k}), \quad (12)$$

$$P_{k+1} = P_{k+1|k} - P_{k+1|k} C^T (C P_{k+1|k} C^T + R)^{-1} C P_{k+1|k}. \quad (13)$$

Optimal Control:

The optimal controller is given by

$$u_k^* = -(U + B^T S B)^{-1} B^T S A \hat{x}_k,$$

where  $S$  is the solution of (9).

## 4.2 UDP case

Information available for the controller at time  $k$ :

$$Y_k = (y_0, \dots, y_k).$$

We do not know whether  $u_k$  has been applied to the system or not. The control actually affect the estimation performance. The optimal control law and the stability of the system is unknown.