

CDS 140a Winter 2014 Homework 6

From MurrayWiki

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(PDF)

WARNING: This homework set is under construction. This banner will be removed when the homework set is finalized.

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. **Perko, Section 3.1, problem 2:** Find the dynamical system associated with the differential equation

$$\dot{x} = x^2$$

and show that it agrees with the result in Example 1 in Section 3.1 of Perko.

- Note: Perko gives all of the components of the answer to this problem; you just need to work through the steps and verify that each piece satisfies the appropriate equations.

2. **Perko, Section 3.1, problem 6, 7:** Two vector fields $f, g \in C^k(\mathbb{R}^n)$ are said to be C^k equivalent on \mathbb{R}^n if there is a homeomorphism $H : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $H, H^{-1} \in C^k(\mathbb{R}^n)$ that maps trajectories of $\dot{x} = f(x)$ to trajectories of $\dot{y} = g(y)$. If $\phi(t)$ and $\psi(t)$ are the dynamics systems defined by f and g respectively, then f and g are equivalent if and only if there exists a strictly increasing function $\tau(x, t)$ such that $\partial\tau/\partial t > 0$ and

$$H(\phi_t(x)) = \psi_{\tau(x,t)}(H(x)).$$

Show the following:

(a) The equilibrium points of $\dot{x} = f(x)$ are mapped to equilibrium points of $\dot{y} = g(y)$.

(b) The eigenvalues of $Df(x_0)$ and the eigenvalues of $Dg(H(x_0))$ differ by the positive multiplicative constant $k_0 = \frac{\partial\tau}{\partial t}(x_0, 0)$.

- Hint: see the problem statements in Perko for some ideas if you get stuck
- This problem can be used to show that the stability of equilibrium points is the same for topologically equivalent systems, since the sign of the real part of the eigenvalues will not be changed.

3. **Perko, Section 3.2, problem 1:** Sketch the phase portrait for the system

$$\begin{aligned}\dot{x} &= x - x^3 \\ \dot{y} &= -y\end{aligned}$$

and show that the interval $[-1, 1]$ on the x -axis is an attracting set for the system. Determine whether this set is an attractor or not and justify your answer.

4. **Perko, Section 3.2, problem 5:**

(a) According to the corollary of Theorem 2 (in Section 3.2), every ω -limit set is an invariant set of the flow ϕ_t of $\dot{x} = f(x)$. Give an example to show that not every set invariant with respect to the flow ϕ_t is the α - or ω -limit set of a trajectory of $\dot{x} = f(x)$.

(b) Any stable limit cycle Γ is an attracting set and Γ is the ω -limit set of every trajectory in a neighborhood of Γ . Give an example to show that not every attracting set A is the ω -limit set of a trajectory in a neighborhood of A .

(c) Is the cylinder in Example 3 of Section 3.2 an attractor for the system in that example?

5. **Perko, Section 3.2, problem 6:** Consider the Lorenz system

$$\begin{aligned}\dot{x} &= \sigma(y - x) \\ \dot{y} &= \rho x - y - xz \\ \dot{z} &= xy - \beta z\end{aligned}$$

with $\sigma, \rho, \beta > 0$.

(a) Show that the system is invariant under the transformation $(x, y, z, t) \rightarrow (-x, -y, z, t)$.

(b) Show that the z -axis is invariant under the flow of the system and that it consists of three trajectories (orbits).

(c) Compute the equilibrium points for the system for the case $\rho > 1$ and show that the equilibrium point at the origin has a one-dimensional unstable manifold $W^u(0)$.

(d) For $\rho \in (0, 1)$, use the Lyapunov function $V(x, y, z) = \rho x^2 + \sigma y^2 + \sigma z^2$ to show that the origin is global asymptotically stable.

6. **Perko, Section 3.3, problem 8:** Consider the system

$$\begin{aligned}\dot{x} &= -y + x(1 - x^2 - y^2)(4 - x^2 - y^2) \\ \dot{y} &= x + y(1 - x^2 - y^2)(4 - x^2 - y^2) \\ \dot{z} &= z.\end{aligned}$$

(a) Show that there are two periodic orbits Γ_1 and Γ_2 in the x, y plane and determine their stability.

(b) Show that there are two invariant cylinders for this system given by $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

(c) Describe $W^s(\Gamma_j)$ and $W^u(\Gamma_j)$, $j = 1, 2$, for the full system (in \mathbb{R}^3).

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