CDS140a
Nonlinear Systems: Local Theory
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# 6 Hamiltonian and Langrangian Formulations

## 6.1 Lagrangian

Often for mechanical systems, one uses the Lagrangian, a function of the "position" and the "velocities" of the mechanical systems

$$L(x_1, \ldots, x_n, v_1, \ldots, v_n).$$

In many cases the Lagrangian is the difference between the potential and the kinetic energy

$$L = K_E - P_E.$$

- *Example 1:* consider a particle moving in $\mathbb{R}^3$ in a potential field $U$. Then $K_E = \frac{1}{2}m\,|v|^2$ and $P_E = V(x)$ (remember that the force is given by $F = -\nabla V$ ) and

$$L = \frac{1}{2}m\,|v|^2 - V(x).$$

The equations of motion are given by the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial v_i} - \frac{\partial L}{\partial x_i} = 0.$$

- In the example this means

$$\begin{aligned} 0 = \frac{d}{dt}\frac{\partial L}{\partial v_i} - \frac{\partial L}{\partial x_i} &= \frac{d}{dt}(mv) + \nabla V \\ &\Downarrow \\ \frac{d}{dt}(mv) &= -\nabla V \end{aligned}$$

which is same as $F = ma$.

*Example 2:* Pendulum. Let $\theta$ be the angle of the pendulum to the vertical, $\dot{\theta}$is the angular velocity, $m$ mass, $l$ pendulum length. Then $K_E = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2$ and $P_E = l - mglcos\theta$,

$$\begin{aligned} L(\theta, \dot{\theta}) &= \frac{1}{2}ml^2\dot{\theta}^2 - l + mgl\cos\theta \\ 0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} &= \frac{d}{dt}\left(ml^2\dot{\theta}\right) + mgl\sin\theta \\ &\Downarrow \\ \ddot{\theta} + \frac{g}{l}\sin\theta &= 0 \end{aligned}$$

The nonlinear harmoic oscillator equation.

## 6.2 Equivalence to Hamiltonian formulation

Let's convert a Lagrangian system into the equivalent Hamiltonian system .

Given the Lagrangian $L(x, v)$, let $p_i = \frac{\partial L}{\partial v_i}$ (called conjugate momentum), then

$$H(x,p) = \sum_i p_i v_i - L(x, v)$$

and the equation of motions are given by

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial x}. \end{aligned}$$

The two formulations (Lagrangian and Hamiltonian) are equivalent.

- Lets check *Example 1* in the subsection above $L = \frac{1}{2}m|v|^2 - V(x)$, $p_i = mv_i$

$$\begin{aligned} H(x,p) &= \sum_i p_i v_i - L(x,v) \\ &= \sum_i m v_i v_i - \frac{1}{2}m|v|^2 + V(x) \\ &= \frac{1}{2}m|v|^2 + V(x) = K_E + P_E \\ &= \frac{1}{2m}|p|^2 + V(x) \end{aligned}$$

  and equations of motion

$$\begin{aligned} \dot{x} &= \frac{p}{m} \\ \dot{p} &= -\nabla V(x). \end{aligned}$$

  i.e., $\frac{d}{dt}p = \frac{d}{dt}(mv) = -\nabla V(x)$.

- Lets check *Example 2* (Pendulum) $L(\theta, \dot{\theta}) = \frac{1}{2}ml^2\dot{\theta}^2 - l + mgl\cos\theta$, $p = ml^2\dot{\theta}$

$$\begin{aligned} H(x,p) &= p\dot{\theta} - L(\theta,\dot{\theta}) \\ &= ml^2\dot{\theta}\dot{\theta} - \frac{1}{2}ml^2\dot{\theta}^2 + l - mgl\cos\theta \\ &= \frac{1}{2}ml^2\dot{\theta}^2 + l - mgl\cos\theta = K_E + P_E \\ &= \frac{1}{2ml^2}p^2 + l - mgl\cos\theta \end{aligned}$$

  and equations of motion

$$\begin{aligned} \frac{d}{dt}\theta &= \frac{p}{ml^2} \qquad (= \dot{\theta}) \\ \dot{p} &= -mgl\sin\theta \end{aligned}$$

  i.e. $\frac{d}{dt}p = \frac{d}{dt}(ml^2\dot{\theta}) = -mgl\sin\theta \Rightarrow \ddot{\theta} + \frac{g}{l}\sin\theta = 0$.

# 7 Summary

Studied (mostly) local properties of

$$\dot{x} = f(x),\ x \in E \subset \mathbb{R}^n \tag{7}$$

1. Existence of Solutions
   (a) First established that if $f \in C^1(E)$ the solutions exist and are unique in a small interval around $t_0$
      i. Build some machinery to be able to prove the result.
      ii. The proof uses successive approximations , one of the basic tools used in the theory of ordinary differential equations. You should be familiar with this proof.
      iii. Similar result can be shown for the non-autonomous system $\dot{x} = f(x,t)$.

   (b) Additionally if $x(0) = y$ then a unique solution exists in a small interval around $t_0$ and for initial conditions in a small neighborhood of $y$.
      i. The solution is continuously differentiable function of $y$ and twice continuously differentiable function of $t$.

   (c) Establish some results about the size of the maximal interval of existence $(\alpha, \beta)$ around $t_0$.
      i. if $\beta$ is finite ($\beta < \infty$), then the solution escapes any compact set (any set that is closed and bounded).
      ii. if the solution is trapped in some compact set, then $\beta = \infty$ (i.e., the solution exists for all positive time)
      iii. If a solutions exists in a closed interval, then solutions that start from nearby initial conditions exist and remain "close" (distance bounded by some exponential of $t$).

2. Flow in a Neigborhood of an Equilibrium Point
   (a) Defined the flow $\phi_t$ of a differential equation by $\phi_t(y) = \phi(t,y)$ where $\phi(t,y)$ is the solution of (7) with initial condition $x(0) = y$.
      i. Can view $\phi_t(y)$ as the motion of a set of initial conditions $y \in K$, i.e., as a mapping of the set $K$ forward (or backward for $t < 0$) in time.
      ii. For a fixed $y$, $\phi_t(y)$ gives a solution trajectory starting at $y$.
      iii. Properties of the flow: $\phi_0(y) = y$; $\phi_s(\phi_t(y)) = \phi_{s+t}(y)$;$\phi_{-t}(\phi_t(y)) = \phi_t(\phi_{-t}(y)) = y$.

   (b) Linearization of (7) around an equilibrium point $x_0$ (i.e., $f(x_0) = 0$) is given by

   $$\dot{x} = Ax \tag{8}$$

   where $A = Df(x_0)$ is the Jacobian evaluated at $x_0$.
      i. If $A$ has no eigenvalues with zero real part, then the Hartman-Grobman theorem allows us to study the local behavior of (7), by looking at the linearized system (8). The two systems (7) and (8) are topologically equivalent near the origin.

   (c) <u>*Inavriant Manifolds.*</u> Let $f(0) = 0$ and rewrite (7) as

   $$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} C & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & Q \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} F(x,y,z) \\ G(x,y,z) \\ H(x,y,z) \end{bmatrix} \tag{9}$$

   where the eigenvalues of $C$ have zero real part, eigenvalues of $P$ have negative real part and eigenvalues of $Q$ have positive rela part, and $(x,y,z) \in \mathbb{R}^c \times \mathbb{R}^s \times \mathbb{R}^u$. The linearization is given by

   $$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} C & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & Q \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{10}$$

   with center, stable and unstable subspaces given by $E^c$, $E^s$, $E^u$.

i. Stable/Center manifold state that there exists center, stable and unstable (invariant) manifolds $W^c(0), W^s(0), W^u(0)$ of (9) tangent to $E^c, E^s, E^u$ respectively. I.e., $W^c(0), W^s(0), W^u(0)$ are to (9) , what $E^c$, $E^s$, $E^u$ are to (10).

ii. Stable manifold theorem provides a way to compute the local stable and unstable manifolds ($S = W^s_{loc}(0)$, $U = W^u_{loc}(0)$ respectively) for *the case when* $c = 0$ (no center subspace).

iii. The calculations allow us to compute $S$ (or $U$) as $S = \{(y, z) \in \mathbb{R}^s \times \mathbb{R}^u \mid z = h(y)\}$ (similarly for $U$). The function $h$ can be thought of as lifting (mapping) $E^s$ into $S$ (i.e., takes a point in $E^s$ and maps it into a point in $S$).

(d) Flow on the center manifold. For the case when there is a center subspace, it can be shown that we can compute the local center manifold

$$W^c_{loc} = \{(x, y, z) \in \mathbb{R}^c \times \mathbb{R}^s \times \mathbb{R}^u \mid y = h_1(x),\ z = h_2(x)\}$$

and the flow on $W^c_{loc}$

$$\dot{x} = Cx + F(x, h_1(x), h_2(x)) \tag{11}$$

(e) Using a Taylor expansion we can calculate approximations of $W^c_{loc}$ and of the corresponding flow on $W^c_{loc}$ .

(f) Because the flow on the stable and unstable manifolds is predictable, *we can essentially determine (locally) the behavior of the* n-dimensional *system* (9) *by studying the behavior of the* c-dimensional *system* (11). In the cases when $c$ is much smaller then $n$ this can be very beneficial.

3. Lyapunov Theory

(a) Defined Lyapunov stable, asymptotically stable, and unstable equilibrium points.

(b) Lyapunov theorem allows to determine the stability of the equilibrium point by using a Lyapunov function $V$ that is positive definite ($V(x) > 0$, $x \neq 0$) and $\dot{V}$ is negative definite ($\dot{V}(x) < 0$, $x \neq 0$) or negative semidefinite ($\dot{V}(x) \leq 0$, $x \neq 0$)

i. Determines stability without explicitly solving the differential equation.

ii. Should know how to prove, at the very least the case when $\dot{V}$ is semidefinite.

iii. Finding the Lyapunov function $V$ is not easy.

iv. If $V$ is radially unbounded ($|x| \to \infty \Rightarrow V(x) \to \infty$), and $\dot{V}$ is negative definite then the origin is globally asymptotic stability.

(c) LaSalle's Invariance principle is a very useful tool to determine system behavior in (positively) invariant compact sets, using a Lyapunov like function.

i. It states that the trajectories that start within the invariant compact set approach the largest invariant subset where $\dot{V}(x) = 0$ as $t \to \infty$

4. Gradient and Hamiltonian Systems

(a) Special and useful cases of nonlinear systems

(b) Defined by twice coontinuously differential potential functions $V(x)$ and Hamiltonian functions $H(x, y)$ respectively.

(c) Nice properties relating the stability of the equilibrium points and the flow of the vector field to the functions $V$ and $H$.