Complexity and fragility in the lattice percolation problem

Maryam Fazel, Xin Liu and John C. Doyle

California Institute of Technology

Connections II, 2006

Outline

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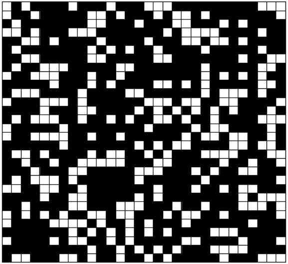
1. Lattice percolation
2. Phase transition
3. Complexity and fragility in lattices
4. Lattice as LP (linear program)

Lattice percolation problem



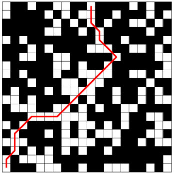
Question

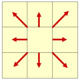
Is there a connected path from the top of the lattice to the bottom through empty sites (a "crash")?

- proper model of a variety of physical systems
- simple, intuitive and easy to visualize
- polynomial time solvable, yet helps develop insights and theory for hard problems; helps understand 'which *instances* are hard'

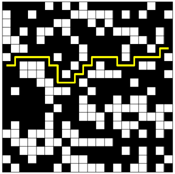
Lattice percolation

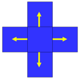
- Vertical path of empties (whites)
- Connect corners or edges
- 8 neighbors





- Horizontal (black) paths
- Connect only on edges
- 4 neighbors

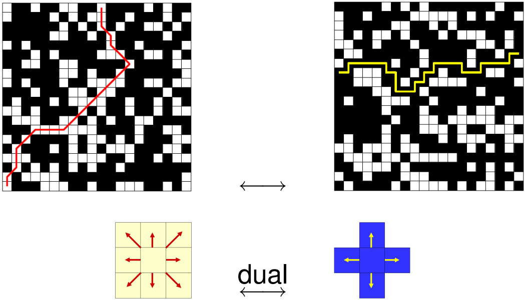




- Assumption: neighborhood rule
- Data: site colorings

Dual rules

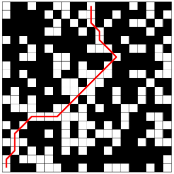
an intuitive notion of duality (details later):



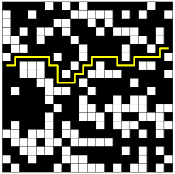
$\{\text{vertical path}\} = \emptyset \iff \{\text{horizontal path}\} \neq \emptyset.$

Paths as proofs

How to prove that “crash” can (cannot) happen?



Crash can happen, as this example shows

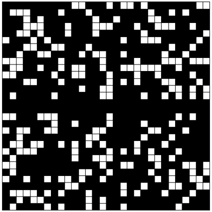


Crash cannot happen as this horizontal path proves

Path length

When the path is long,

- finding it tends to be hard (you as the“computer”)
- describing it is hard



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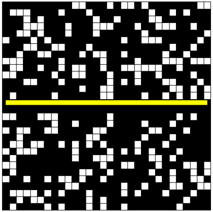

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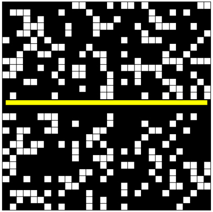

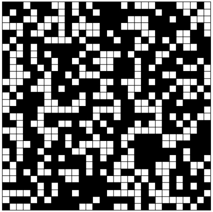

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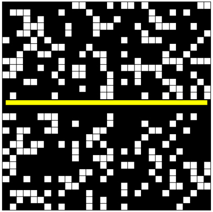

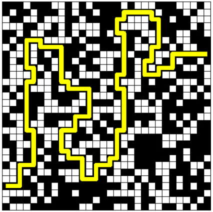

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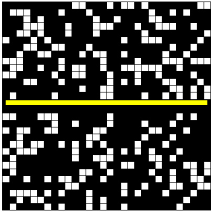
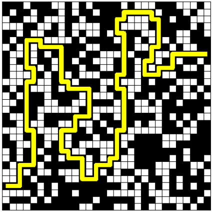
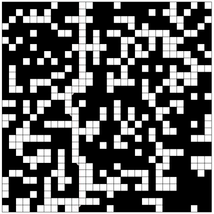

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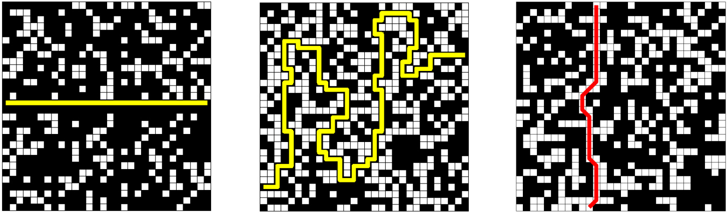

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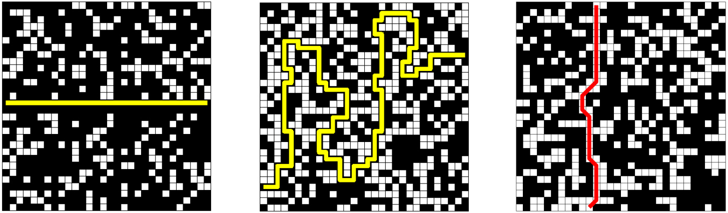
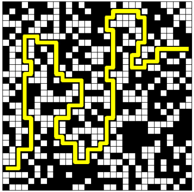


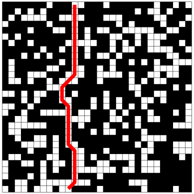
Intuition

Path length can represent proof complexity. . .

Outline

Phase transition



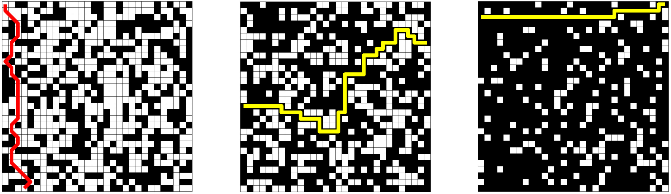


$\rho = 0.59$

Phase transition

- considers random lattices
- is thought to be linked with complex cases, where paths are long
- but long proof and critical density do no always happen together.

Phase transition



$\rho = 0.4$ $\rho = 0.6$ $\rho = 0.8$

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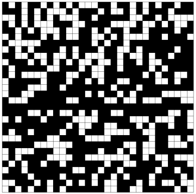
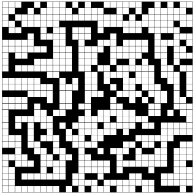
Phase transition



$\rho = 0.61$



$\rho = 0.31$

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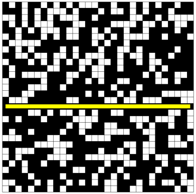

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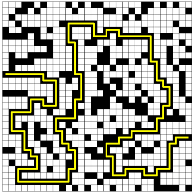

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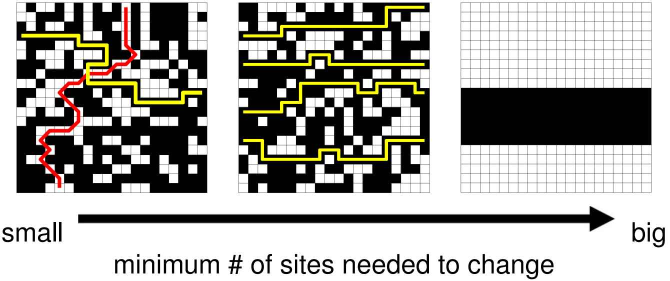
What is fragility

Robustness?

what is the smallest change in problem data to change the answer?



Complexity and fragility

Definitions

ρ =density of occupied sites; n =size of lattice;

ℓ =length of shortest path; $C = \dfrac{\ell}{n}$;

b =number of independent paths; $F = \dfrac{\rho n}{b}$.

Conjecture

$$C \leq F$$

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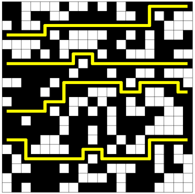


Simple proof:

$$\ell b \leq \rho n^2 \Rightarrow \frac{\ell}{n} \leq \frac{\rho n}{b} \Rightarrow C \leq F.$$

we can build lattices that show the bound is tight.

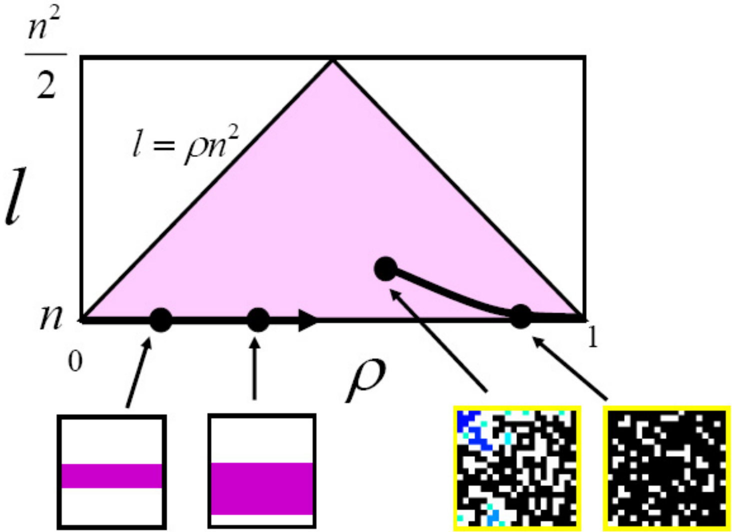
2D vs higher dimensions

2D lattices are special:

- primal and dual problems are essentially the same
- dual of paths are paths
- there is no duality gap

- in higher dimensions, e.g., 3: dual of a path is a surface
- in general, barrier that stops a 1D path in an n-D lattice is $n-1$ dimensional
- neighborhood rules generalize

Complexity and fragility



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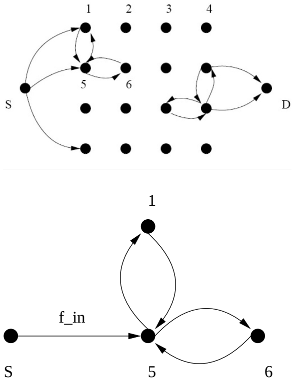
Lattice as LP



Flow model for lattice

- write flow conservation for all nodes, e.g., node 5:

$$-f_{15} - f_{65} + f_{51} + f_{56} - f_{\text{in}} = 0$$

- to check if path exists: find f such that

$$Af = b, \quad f \succeq 0,$$

f = vector of all flows f_{ij},
A = incidence matrix,
b = source/destination flows

Farkas' Lemma

Farkas' lemma

The following two systems

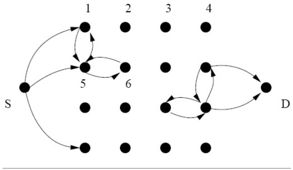
$$Ax \preceq 0,\ c^T x < 0 \quad \text{and} \quad A^T y + c = 0,\ y \succeq 0$$

where $A \in \mathbb{R}^{m \times n}$ and $c \in \mathbb{R}^n$, are *strong alternatives*; i.e., one and only one is true.

Applying Farkas' lemma to $Af = b, \quad f \succeq 0$ gives **alternative (dual) LP:**

$$A^T \nu \succeq 0, \quad b^T \nu < 0.$$

Dual variables and barrier



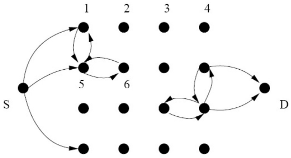
Interpretation of dual variables ν

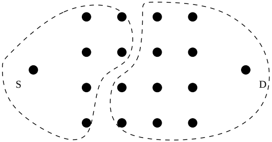
- alternative problem:
  $A^T\nu \succeq 0, \quad b^T\nu < 0.$
- reduces to:

$$\nu_i - \nu_j \geq 0 \text{ if } i \to j,$$
$$\nu_D - \nu_S < 0,$$

- for all nodes (except S, D) flows are bi-directional, yielding equal ν for all connected nodes.
- νs can be used to indicate disconnected clusters.

Dual variables and barrier





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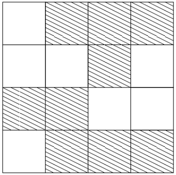
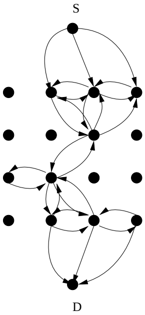
Dual variables and barrier

Idea: tracing the break

finding ν is ‘equivalent’ to finding a vertical path with 8-neighbor rule in the dual lattice.





Lattice duality can be viewed as a special case of LP duality.

Shortest path

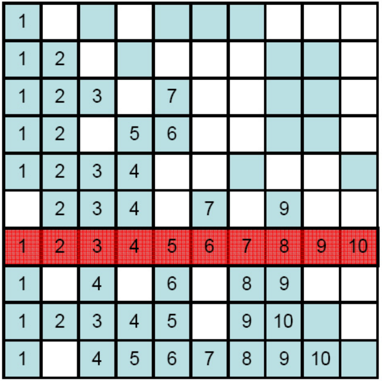
Finding shortest path

$$\begin{array}{ll} \text{minimize} & (\#\text{ of non-zero } f_{ij}) \\ \text{subject to} & Af = b, \\ & f \succeq 0. \end{array}$$

due to special property of A, b (total unimodularity), reduces to

$$\begin{array}{ll} \text{minimize} & \sum_{ij} f_{ij} \\ \text{subject to} & Af = b, \\ & f \succeq 0. \end{array}$$

Shortest path



- Solving LP directly is not an efficient way to check for shortest path. Breadth-first search is better.
- BFS runtime related to shortest path length

Summary

- lattices (visually) illustrate key issues of duality and complexity
- random cases well-studied, e.g., phase transition
- lattice duality is a special case of LP duality
  - do the insights extend to general LPs?
- Complexity implies Fragility