CDS 202 Winter 2009 Solution Set 5

Problem 1 (Ramon van Handel)

We are given the vector fields on $\mathbb{R}^3$ ($x = (x^1, x^2, x^3) \in \mathbb{R}^3$)

$$X(x) = \frac{\partial}{\partial x^2} - x^1 \frac{\partial}{\partial x^3}$$
$$Y(x) = \frac{\partial}{\partial x^1}\,.$$

We easily calculate by commutation

$$[X, Y] = \left(\frac{\partial}{\partial x^2} - x^1 \frac{\partial}{\partial x^3}\right)\frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^1}\left(\frac{\partial}{\partial x^2} - x^1 \frac{\partial}{\partial x^3}\right) = \frac{\partial}{\partial x^3}\,.$$

Next, we calculate the flow for each of these vector fields. This amounts to solving the corresponding set of differential equations. For X we have

$$\dot{x} = (0, 1, -x^1) \implies x(t) = (x_0^1, x_0^2 + t, x_0^3 - x_0^1 t) = \phi_t^X(x_0)\,.$$

For Y we get

$$\dot{x} = (1, 0, 0) \implies x(t) = (x_0^1 + t, x_0^2, x_0^3) = \phi_t^Y(x_0)\,.$$

and for $[X, Y]$ we get

$$\dot{x} = (0, 0, 1) \implies x(t) = (x_0^1, x_0^2, x_0^3 + t) = \phi_t^{[X,Y]}(x_0)\,.$$

We can now evaluate the expression in the problem. The RHS gives

$$\phi_{h^2}^{[X,Y]}((0, 0, 0)) = (0, 0, h^2)\,.$$

Evaluate the LHS:

$$\phi_h^X((0, 0, 0)) = (0, h, 0)\phi_h^Y \circ \phi_h^X((0, 0, 0)) = \phi_h^Y((0, h, 0)) = (h, h, 0)\,.$$

Now use $\phi_h^{-X} = \phi_{-h}^X$, $\phi_h^{-Y} = \phi_{-h}^Y$ (as $dx(t)/dt = X(x)$, so $dx(-t)/dt = -X(x)$.) We get

$$\phi_h^{-Y} \circ \phi_h^{-X} \circ \phi_h^Y \circ \phi_h^X((0, 0, 0)) = \phi_{-h}^Y \circ \phi_{-h}^X((h, h, 0)) = \phi_{-h}^Y((h, 0, h^2)) = (0, 0, h^2)\,.$$

Thus $\phi_h^{-Y} \circ \phi_h^{-X} \circ \phi_h^Y \circ \phi_h^X((0, 0, 0)) = \phi_{h^2}^{[X,Y]}((0, 0, 0))$.

Problem 2 (Nawaf Bou-Rabee)

Since $X, Y \in \Delta$, one can expand them in terms of the basis elements of the distribution:

$$X = \sum_{i=1}^{d} f_i X_i$$
$$Y = \sum_{i=1}^{d} g_i X_i$$

where $f_i, g_i \in C^\infty(M)$ for $i = 1, ..., d$. Therefore,

$$\begin{aligned}
[X, Y] &= [\sum_{i=1}^{d} f_i X_i, \sum_{j=1}^{d} g_j X_j] \\
&= \sum_{i=1}^{d} \sum_{j=1}^{d} [f_i X_i, g_j X_j] \\
&= \sum_{i=1}^{d} \sum_{j=1}^{d} f_i g_j [X_i, X_j] + f_i(X_i g_j) X_j - g_j(X_j f_i) X_i
\end{aligned}$$

Since $f_i g_j [X_i, X_j] \in \Delta$, $f_i(X_i g_j) X_j \in \Delta$, and $g_j(X_j f_i) X_i \in \Delta$, then $[f_i X_i, g_j X_j] \in \Delta$. Finally, $[X, Y] \in \Delta$ since it is the sum of $[f_i X_i, g_j X_j]$.

Problem 3 (PS 2009)

Quick proof: $X_p \in T_p N$ $\forall p \in N$ means there exists $X^N \in \mathfrak{X}(N)$ such that $X^N \sim_i X$, where $i : N \to M$ is the natural inclusion. Similarly for $Y^N \sim_i Y$. Then Prop 4.2.25$\Rightarrow$ $[X^N, Y^N] \sim_i [X, Y]$ i.e. $[X, Y]_p \in T_p N$ $\forall p \in N$.

Equivalent (but longer) proof: $X_p \in T_p N$ $\forall p \in N$ means there exists $X^N \in \mathfrak{X}(N)$ such that

$$(Xf)|_N = X^N(f|_N) \quad \forall f \in C^\infty(M).$$

Similarly $(Yf)|_N = Y^N(f|_N)$ for some $Y^N \in \mathfrak{X}(N)$. Then

$$(X(Yf))|_N = X^N((Yf)|_N) = X^N(Y^N(f|_N))$$

and similarly for $Y(Xf))$. So

$$([X, Y]f)|_N = [X^N, Y^N](f|_N) \quad \forall f \in C^\infty(M).$$

By definition this means $[X, Y]_p \in T_p N$ $\forall p \in N$.

Problem 4 (PS 2009)

Since F is a submersion, for every point $m \in M$, $T_m F$ is surjective with split kernel. Then by the Local Onto Theorem (Thm 3.5.2), there exist charts (U, ϕ) and (V, ψ) such that $m \in U$, $f(U) \subset V$, $\phi : U \to \phi(U) = U' \times V'$, and $F_{\phi\psi} : U' \times V' \to V'$ is projection onto the second factor.

The sets $F^{-1}(q),\ q \in N$ clearly partition M. Construct the above charts for $m \in M$. We have that

$$\begin{aligned} p \in U \cap F^{-1}(q) &\Leftrightarrow p \in U \text{ and } F(p) = q \\ &\Leftrightarrow p \in U \text{ and } \psi(F(p)) = \psi(q) \\ &\Leftrightarrow p \in U \text{ and } F_{\phi\psi}(\phi(p)) = \psi(q) \\ &\Leftrightarrow \phi(p) \in U' \times V' \text{ and } \phi(p) \in \mathbf{E} \times \{\psi(q)\} \\ &\Leftrightarrow \phi(p) \in U' \times \{\psi(q)\} \end{aligned}$$

So $\phi(U \cap F^{-1}(q)) = U' \times \{\psi(q)\}$, as required. Since we can do this for any point $m \in M$, this proves that the sets $F^{-1}(n)$ are leaves of a foliation (assuming they are connected).

Problem 5 (PS 2009)

(a) The flow of X through (x, y, θ) is the solution of $\gamma'(t) = X(\gamma(t)),\ \gamma(0) = (x, y, \theta)$, and so is given by

$$\phi_t^X(x, y, \theta) = (x + t\cos\theta, y + t\sin\theta, \theta).$$

Similarly

$$\phi_t^Y(x, y, \theta) = (x, y, \theta + t).$$

The flows are complete since they are defined for all t.

(b) X is invariant under its own flow iff $(\phi_t^X)_* X = X$ i.e.

$$T_{\phi_{-t}^X(p)}\phi_t^X(X_{\phi_{-t}^X(p)}) = X_p \qquad \forall p \in M.$$

The components of $T_{\phi_{-t}^X(p)}\phi_t^X(X_{\phi_{-t}^X(p)})$ wrt the basis $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta}\right\}\Big|_p$ are

$$\begin{pmatrix} 1 & 0 & -t\sin\theta \\ 0 & 1 & t\cos\theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$

which is the same as the components of X_p wrt this basis. Similarly, to show $T_{\phi_{-t}^Y(p)}\phi_t^Y(Y_{\phi_{-t}^Y(p)}) = Y_p$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

which is the same as the components of Y_p.

(c)

$$\begin{aligned}[X,Y] &= \left(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\right)\frac{\partial}{\partial\theta} - \frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\right)\\ &= \sin\theta\frac{\partial}{\partial x} - \cos\theta\frac{\partial}{\partial y}.\end{aligned}$$

Since

$$\det\begin{pmatrix}\sin\theta & -\cos\theta & 0\\ \cos\theta & \sin\theta & 0\\ 0 & 0 & 1\end{pmatrix} = 1 \neq 0$$

the vectors $X_p, Y_p,$ and $[X,Y]_p$ are linearly independent for each $p \in M$, and so span the three dimensional space T_pM.

(d)

$$\begin{aligned}&\phi(x,y,\theta) = (x\cos\theta + y\sin\theta, x\sin\theta - y\cos\theta, \theta)\\ \Rightarrow&\phi^{-1}(z_1,z_2,z_3) = (z_1\cos z_3 + z_2\sin z_3, z_1\sin z_3 - z_2\cos z_3, z_3).\end{aligned}$$

(**Note:** since ϕ is a change of coordinates, we're really just computing the components of X wrt two different coordinate systems; it's not really correct to say this is the 'push-forward' of X by ϕ.)

The components $\widetilde{X}^i$ of X wrt the basis $\{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}\}$ are related to the components X^j wrt the basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial\theta}\}$ by

$$\widetilde{X}^i(z) = \frac{\partial\phi^i}{\partial x^j}(\phi^{-1}(z))X^j(\phi^{-1}(z))$$

so the $\widetilde{X}^i$ are given by

$$\begin{aligned}&\begin{pmatrix}\cos z_3 & \sin z_3 & -(z_1\cos z_3 + z_2\sin z_3)\sin z_3 + (z_1\sin z_3 - z_2\cos z_3)\cos z_3\\ \sin z_3 & -\cos z_3 & (z_1\cos z_3 + z_2\sin z_3)\cos z_3 + (z_1\sin z_3 - z_2\cos z_3)\sin z_3\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos z_3\\ \sin z_3\\ 0\end{pmatrix}\\ &= \begin{pmatrix}\cos z_3 & \sin z_3 & -z_2\\ \sin z_3 & -\cos z_3 & z_1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos z_3\\ \sin z_3\\ 0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}\end{aligned}$$

i.e. $X = \frac{\partial}{\partial z_1}$. Similarly, the $\widetilde{Y}^i$ are given by

$$\begin{pmatrix}\cos z_3 & \sin z_3 & -z_2\\ \sin z_3 & -\cos z_3 & z_1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}-z_2\\ z_1\\ 1\end{pmatrix}$$

i.e. $Y = -z_2\frac{\partial}{\partial z_1} + z_1\frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3}$.

(e) Components $\widetilde{[X,Y]}^i$ of $[X,Y]$ are given by

$$\begin{pmatrix} \cos z_3 & \sin z_3 & -z_2 \\ \sin z_3 & -\cos z_3 & z_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sin z_3 \\ -\cos z_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

i.e. $[X,Y] = \frac{\partial}{\partial z_2}$. Computing the other way,

$$\begin{aligned} [X,Y] &= \left[\frac{\partial}{\partial z_1},\ -z_2\frac{\partial}{\partial z_1} + z_1\frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3}\right] \\ &= \frac{\partial}{\partial z_2}. \end{aligned}$$

So $[X,Y]$ works out the same either way we calculate it.

(f) The invariance condition $F_* X = X$ is equivalent to

$$T_{F^{-1}(p)}F(X_{F^{-1}(p)}) = X(p) \qquad \forall p \in M,$$

and similarly for Y. The components of $T_pF(X_{F^{-1}(p)})$ with respect to $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta}\}\big|_p$ are

$$\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta-\alpha) \\ \sin(\theta-\alpha) \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}.$$

We can see these are the same as the components of X_p wrt $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta}\}\big|_p$.