CDS 140a Winter 2014 Homework 5

From MurrayWiki

R. Murray, D. MacMartin — Issued: 4 Feb 2013 (Tue)

ACM 101b/AM 125b/CDS 140a, Winter 2014 — Due: 12 Feb 2013 (Wed) @ noon

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. **Perko, Section 2.9, problem 3** Use the Lyapunov function $V(x) = x_1^2 + x_2^2 + x_3^2$ to show that the origin is an asymptotically stable equilibrium point of the system
$$\dot{x} = \begin{bmatrix} -x_2 - x_1 x_2^2 + x_3^2 - x_1^3 \\ x_1 + x_3^3 - x_2^3 \\ -x_1 x_3 - x_3 x_1^2 - x_2 x_3^2 - x_3^5 \end{bmatrix}$$
Show that the trajectories of the linearized system $\dot{x} = Df(0)x$ for this problem lie on circles in planes parallel to the x_1, x_2 plane; hence, the origin is stable, but not asymptotically stable for the linearized system.

2. Determine the stability of the system
$$\dot{x} = -y - x^3$$
$$\dot{y} = x^5$$
Motivated by the first equation, try a Lyapunov function of the form $V(x, y) = \alpha x^6 + \beta y^2$. Is the origin asymptotically stable? Is the origin globally asymptotically stable?

3. Definition: An equilibrium point is *exponentially stable* if $\exists M, \alpha > 0$ and $\epsilon > 0$ such that $\|x(t)\| \leq Me^{-\alpha t}\|x(0)\|, \forall \|x(0)\| \leq \epsilon,\ t \geq 0$ Let $\dot{x} = f(x)$ be a dynamical system with an equilibrium point at $x_e = 0$. Show that if there is a function $V(x)$ satisfying
$$k_1\|x\|^2 \leq V(x) \leq k_2\|x\|^2, \quad \dot{V}(x) \leq -k_3\|x\|^2$$
for positive constants k_1, k_2 and k_3, then the equilibrium point at the origin is exponentially stable.
   - Hint: you can use Gronwall's inequality from Section 2.3 of Perko.

4. **Perko, Section 2.12, problem 2** Use Theorem 1 [Center Manifold Theorem] to determine the qualitative behaviour near the non-hyperbolic critical point at the origin for the system
$$\dot{x} = y$$
$$\dot{y} = -y + \alpha x^2 + xy$$
for $\alpha \neq 0$ and for $\alpha = 0$; i.e., follow the procedure in Example 1 after diagonalizing the system as in Example 3.

5. Consider the following system in $\mathbb{R}^2$:
$$\dot{x} = -\frac{\alpha}{2}(x^2 + y^2) + \alpha(x + y) - \alpha$$
$$\dot{y} = -\alpha xy + \alpha(x + y) - \alpha$$

Determine the stable, unstable, and centre manifold of the equilibrium point at $(x, y) = (1, 1)$, and determine the stability of this equilibrium point for $\alpha \neq 0$. For determining stability, note that near the equilibrium point there are two 1-dimensional invariant linear manifolds of the form $M = \{(a_1, a_2) \in \mathbb{R}^2 \| a_2 = ka_1\}$; determine the flow on these invariant manifolds.