Outline

1. **The Logistic Map**
   - Invariance Proofs
   - Proof Length (”Complexity”)
2. **Mandelbrot Set**
   - Inner and Outer Bounds
   - Fragility in the Mandelbrot Set
3. **Summary**

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1. **The Logistic Map**
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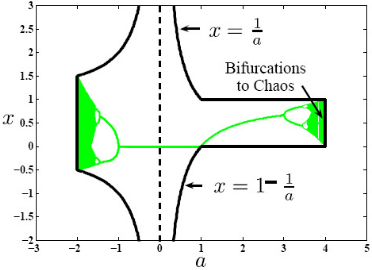
   - Proof Length ("Complexity")
2. **Mandelbrot Set**
   - Inner and Outer Bounds
   - Fragility in the Mandelbrot Set
3. **Summary**

What is the Logistic Map?

Defining Equation

$$x_{k+1} = ax_k(1 - x_k)$$
$$\textit{where} \quad a, x_k \in \mathbb{R}$$



- Complex dynamics and bifurcation to chaos
- Allows us to visualize dynamics vs. parameter in 2D
- Fixed points at $x = 0$ and $x = 1 - \frac{1}{a}$
- Bifurcation occurs at $a = 1$ (fixed points interchange stability properties)

Region of Attraction

$$\begin{array}{cc} 0 \leq x \leq 1 & 1 \leq a \leq 4 \\ 1 - \dfrac{1}{a} \leq x \leq \dfrac{1}{a} & 0 \leq a \leq 1 \\ \dfrac{1}{a} \leq x \leq 1 - \dfrac{1}{a} & -2 \leq a \leq 0. \end{array}$$

Breaking the region of Attraction into Branches

- Any semialgebraic set can be written as a union of basic semialgebraic sets.
- Proof can always be broken into pieces (union of empty sets obviously empty).
- Technique for breaking proofs into sets is reminiscent of branch and bound in optimization.
- In this case 2 sets is natural based on the geometry.
- In general figuring out how to do this is not easy and choosing wrong affects the proof "length".

Semialgebraic sets for the branches

$$\{1-(2x-1)^2 \geq 0; -(a-1)(a-4) \geq 0\}$$
$$\{(a-2)^2 - a^2(2x-1)^2 \geq 0; -(a+2)(a-1) \geq 0\}$$

Definitions

Definition

Given polynomials $\{g_1, \ldots, g_t\} \in \mathbb{R}[\mathbf{x}]$ the *Multiplicative Monoid* generated by the g_j's is the set of all finite products of the g_j's including 1. This will be denoted by $\mathbf{M}(g_1, \ldots, g_t)$

Definition

Given polynomials $\{f_1, \ldots, f_s\} \in \mathbb{R}[\mathbf{x}]$ the *Algebraic Cone* generated by the f_i's is the set

$$\mathbf{C}(f_1, \ldots, f_s) = \left\{ f \,\middle|\, f = \lambda_0 + \sum_i \lambda_i F_i \right\}$$

where $F_i \in \mathbf{M}(f_1, \ldots, f_s)$, λ_i's are SOS Polynomials

Definition

Given polynomials $\{h_1, \ldots, h_r\} \in \mathbb{R}[\mathbf{x}]$ the *Ideal* generated by the h_k's is the set

$$\mathbf{I}(h_1, \ldots, h_r) := \left\{ h \,\middle|\, h = \sum_k \mu_k h_k \right\} \quad \textit{where} \quad \mu_k \in \mathbb{R}[\mathbf{x}]$$

Positivstellensatz

Theorem

The set $\quad \{f_i(\mathbf{x}) \geq 0, \quad g_j(\mathbf{x}) \neq 0, \quad h_k(\mathbf{x}) = 0\}$

is infeasible in $\mathbb{R}^n$ *if and only if* $\exists\ F,\ G,\ H$ *such that*

$$H + F = -G^2$$

where $\quad i = 1, \ldots, s \;\; j = 1, \ldots, t \;\; k = 1, \ldots, r$

$F \in \mathbf{C}(f_1, \ldots, f_s),\ G \in \mathbf{M}(g_1, \ldots, g_t),\ H \in \mathbf{I}(h_1, \ldots, h_r)$

- Holds for arbitrary systems of polynomial equations, non-equalities and inequalities over the reals
- By construction $H + F \geq 0$ so $H + F = -G^2$ provides a contradiction.
- Proofs called infeasibility certificates (P-satz refutations)

Useful expressions

Definition

The *subset of the cone* is the set of F_i's in the definition of the **Cone**.

Definition

The *proof order* is the degree of the highest order term in the Positivstellensatz refutation.

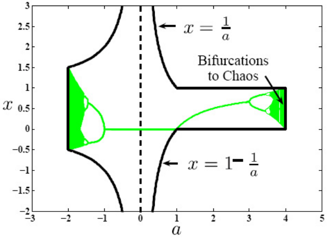
Definition

The *SOS multiplier order* is the order of each of the λ_i's in the **Cone**.

Branch 1: $-2 \leq a \leq 1$

The Constraint Set

$$f_1(a,x) = (a-2)^2 - a^2(2x-1)^2 \geq 0$$
$$f_2(a,x) = -(a+2)(a-1) \geq 0$$
$$f_3(a,x) = a^2(2ax(1-x)-1)^2 - (a-2)^2 \geq 0$$
$$f_3(a,x) \neq 0$$



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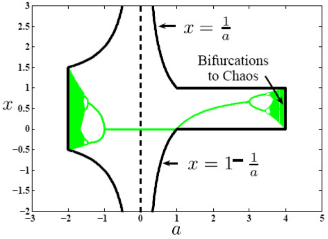

Want SOS polynomials $p_0,\ p_i,\ p_{ij},\ p_{ijk}$

$$-(f_3^\alpha)^2 = p_0 + \sum_i p_i f_i + \sum_{\{i,j\}} p_{ij} f_i f_j + \sum_{\{i,j,k\}} p_{ijk} f_i f_j f_k \qquad \alpha \in \{0,1,2\dots\}$$

Branch 1: $-2 \leq a \leq 1$

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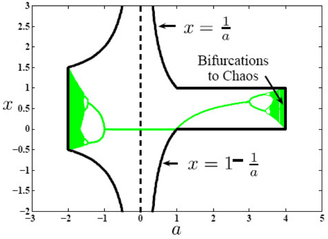


Form of the Refutation

$$-f_3^2 = p_{13} f_1 f_3 + p_{123} f_1 f_2 f_3$$

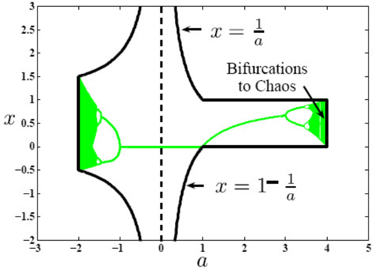
where $p_{13}(a,x) = \frac{4}{3} - \frac{2}{3}a + \frac{1}{3}a^2 - xa^2 + x^2a^2, \quad p_{123}(a,x) = \frac{1}{3}.$

Note $\quad p_{13} = \dfrac{1}{3} f_2 + a(x^2 a - ax + 1) \quad$ *and* $\quad f_3 = -a(ax^2 - xa + 1) f_1$

Branch 2: $1 \le a \le 4$

The Constraint Set

$$f_1(a,x) = 1 - (2x-1)^2 \ge 0$$
$$f_2(a,x) = -(a-1)(a-4) \ge 0$$
$$f_3(a,x) = (2ax(1-x)-1)^2 - 1 \ge 0$$
$$f_3(a,x) \ne 0.$$



Branch 2: $1 \leq a \leq 4$

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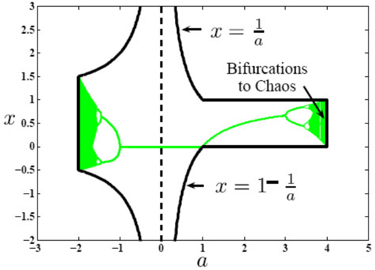

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Note $\quad p_{13} = -\dfrac{1}{3} f_2 + (a^2x^2 - xa^2 + 1) \quad$ *and* $\quad f_3 = -f_1(a^2x^2 - xa^2 + 1)$

How to define/classify ‘Proof Length’

- Order of the Proof and/or Order of the SOS Multipliers
- Size and Conditioning of the SDP

Example (Order of the Proof)

For the $1 \leq x \leq 4$ an alternative refutation can is:

$$-f_3^2 = p_0 + p_1 f_1 + p_2 f_2 + p_3 f_3$$

Polynomial	Order in x	Order in a
p_0	8	4
p_1	6	4
p_2	8	2
p_3	4	2

- Proofs same order but use different subsets of the cone.
- This proof is linear in f_i's but the SOS multipliers more complicated.
- Which proof is longer?
- Size and Conditioning of the SDP may be a more natural choice BUT are implementation dependent!

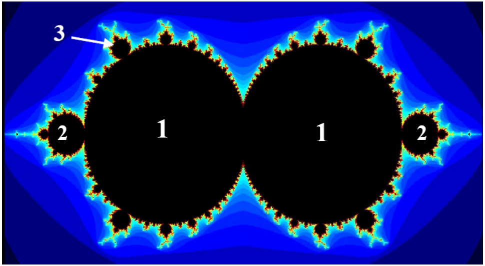
Outline

What is the Mandelbrot Set?

The λ Parameterization

$$z_{k+1} = \lambda z_k(1 - z_k)$$

$$\textit{where} \quad \lambda, z_k \in \mathbb{C}$$



- The complex version of the logistic map
- Fixed points at $z = 0$ and $z = 1 - \frac{1}{\lambda}$
- $\lambda \in \textit{Mset} \;\Leftrightarrow\; z_k$ *bounded*
- Color indicates no. iterations to unboundedness (interpretation “distance” from Mset)
- Important to note that Mandelbrot set is a subset of parameter space not dynamical system space

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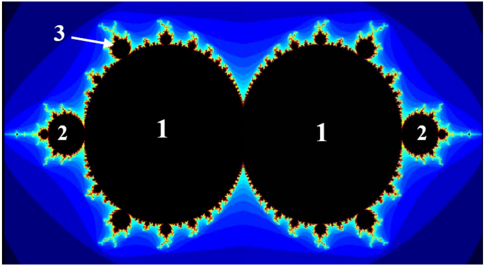
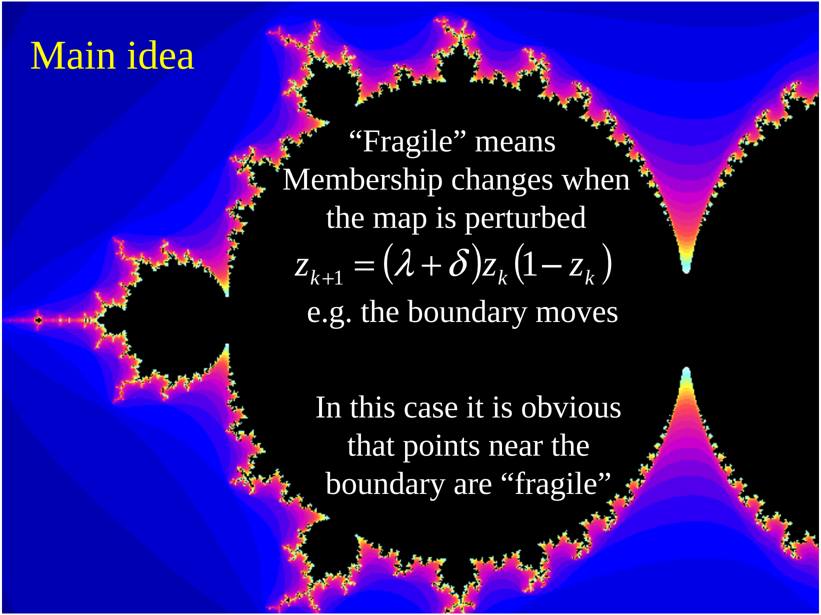


- **Set membership is undecidable in the sense of Turing**
- Classic computational problem that is easily visualized.
- Most computational problems involve uncertain dynamical systems, from protein folding to complex network analysis. Not easily visualized.
- Natural questions are typically computationally intractable, and conventional methods provide little encouragement that this can be systematically overcome.

Fragility In the Mandelbrot Set



Cyclic Lobes: Regional (“Global”) Proofs

$$z_{k+1} = \lambda z_k(1 - z_k)$$

$$V(z_k) = |z_k|^2$$

$$\textit{Stability} \quad \Leftrightarrow V(z_k) \geq V(z_{k+1})$$

$$\Leftrightarrow |z_k|^2 - |\lambda z_k(1 - z_k)|^2 \geq 0$$

$$\Leftarrow 1 \geq |\lambda||(1 - z_k)|$$

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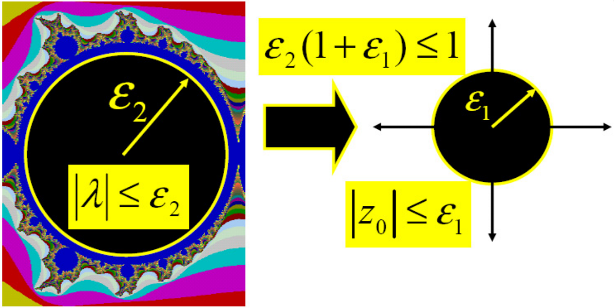

$$\{\lambda \leq \mathbf{1}\} \subset \textit{Mset}$$

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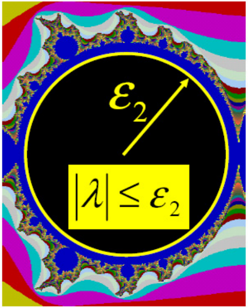
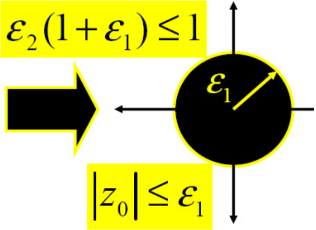
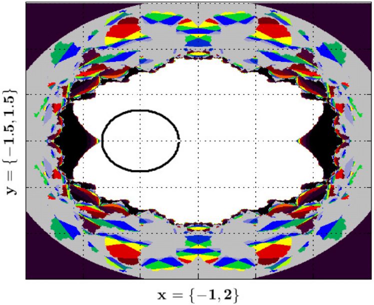




Julia Sets for $|\lambda| = 0.75$



$$\{\lambda \leq 1\} \subset \textit{ Mset}$$

The Left Lobe

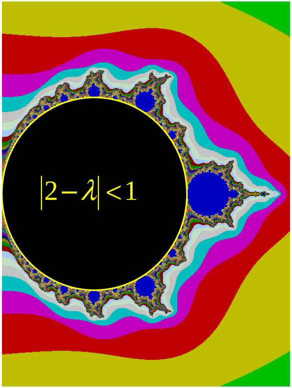
Fixed point at $z = (1 - \frac{1}{\lambda})$

$$\text{let} \quad w_k = z_k - z^* \quad \text{then}$$
$$w_{k+1} = w_k(2 - \lambda - \lambda w_k)$$

Using a similar Lyapunov Function

$$V(w_k) = |w_k|^2$$

$$|w_{k+1}|^2 \leq |w_k|^2 \Leftarrow |2-\lambda| + |\lambda||w_k| \leq 1$$



$\{|2 - \lambda| \leq 1\} \subset \; Mset$

Regional ('Global') in λ Local in z

- The 2-period map is

$$\begin{aligned} Q(z) = z_{k+2} &= \lambda z_{k+1}(1 - z_{k+1}) \\ &= \lambda^2 z_k(1 - z_k)(1 - \lambda z_k + \lambda z_k^2) \end{aligned}$$

- The fixed points of this map are

$$z_1^* = 0, z_2^* = 1 - \frac{1}{\lambda}$$

$$z_{3,4}^* = \frac{\lambda + 1 \pm \sqrt{\lambda^2 - 2\lambda - 3}}{2\lambda}$$

For an attracting fixed point

$$\left|\dot{Q}\right| < 1$$

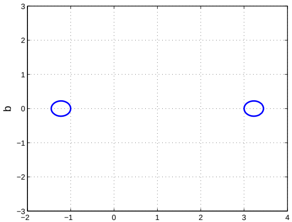
Using z_3^*

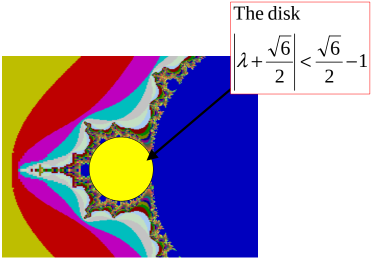
$$\begin{aligned} \dot{Q}(z_3^*) &= \frac{d}{dx} F(F(z))_{z=z_3^*} \\ &= F'(F(z_3^*))F'(z_3^*) \\ &= F'(z_4^*)F'(z_3^*) \\ &= 4 + 2\lambda - \lambda^2 \end{aligned}$$

Therefore the 2-cycle is locally attracting for $|4 + 2\lambda - \lambda^2| < 1$.

2 Period Lobes

Letting $\lambda = a + bi$ gives
$(4 - a^2 + 2a + b)^2 + (2b - 2ab)^2 < 1$.





Want to show that:

$$\left|\lambda + \frac{\sqrt{6}}{2}\right| < \frac{\sqrt{6}}{2} - 1 \subseteq (4 - a^2 + 2a + b)^2 + (2b - 2ab)^2 - 1 < 0$$

This is equivalent to showing that;

$$\left\{ \begin{array}{c} (4 - a^2 + 2a + b)^2 + (2b - 2ab)^2 - 1 \geq 0 \\ \left(\frac{\sqrt{6}}{2} - 1\right)^2 - \left(a + \frac{\sqrt{6}}{2}\right)^2 + b^2 > 0 \end{array} \right\} = \varnothing$$

2 Period Lobes

Constraint Set

$$\begin{aligned} f_1 &= (4 - a^2 + 2a + b)^2 + (2b - 2ab)^2 - 1 \geq 0 \\ f_2 &= \left(\frac{\sqrt{6}}{2} - 1\right)^2 - \left(a + \frac{\sqrt{6}}{2}\right)^2 - b^2 - \varepsilon \geq 0 \end{aligned}$$

Positivstellensatz refutation

$$p_0 + p_1 f_1 + p_2 f_2 = -1$$

$p_1 \simeq 395$ and $p_2 = 4465.4 + 667.03a^2 - 1974.1a + 1223.3b^2$

- Determining set membership for local z values in the two period region required an increase in both the order and the size of the proof.
- The proof is also ill conditioned.
- These differences are associated with an increase in proof length or 'complexity'.

2 Period Lobes

Constraint Set

$$
\begin{aligned}
f_1 &= (4 - a^2 + 2a + b)^2 + (2b - 2ab)^2 - 1 \geq 0 \\
f_2 &= \left(\frac{\sqrt{6}}{2} - 1\right)^2 - \left(a + \frac{\sqrt{6}}{2}\right)^2 - b^2 - \varepsilon \geq 0
\end{aligned}
$$

Positivstellensatz refutation (increasing ε)

$$p_0 + p_1 f_1 + p_2 f_2 = -1$$

$p_1 = 19.51$ and $p_2 = 223.48 + 49.25a^2 - 112.24a + 68.32b^2$

- Moving further away from the boundary (less fragile region) improves conditioning.
- This is good evidence that SDP conditioning should be part of proof length definition.

2 Period Lobes

Constraint Set

$$\begin{aligned} f_1 &= (4 - a^2 + 2a + b)^2 + (2b - 2ab)^2 - 1 \geq 0 \\ f_2 &= \left(\frac{\sqrt{6}}{2} - 1\right)^2 - \left(a + \frac{\sqrt{6}}{2}\right)^2 - b^2 - \varepsilon \geq 0 \end{aligned}$$

Positivstellensatz refutation 2 (setting $\varepsilon = 0$)

$$p_0 + p_1 f_1 + p_3 f_1 f_2 = -f2^2$$

$$p_2 = 1.4b^4 + a^4 + 4.8a^3 + 2.9a^2b^2 + 7ab^2 + 8.6a^2 + 6.9a + 4.1b^2 + 2$$
$$p_3 = 1.2a^2b^2 - .4ab^2 + .3a^2 + .94b^2 + .35a + .34$$

- Higher proof order with better conditioning

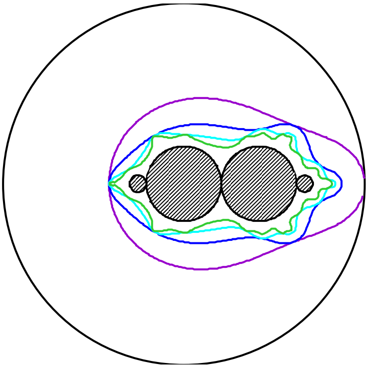
Outer Bounds

Assume

$\lambda \notin \{|\lambda| \leq 1\} \cup \{|\lambda - 2| \leq 1\}$

$V(z_k) = |z_k|^2$ increases

$$\begin{aligned} & V(z_k) \leq V(z_{k+1}) \\ \Leftrightarrow\ & 1 \leq |\lambda||(1 - z_k)| \\ \Leftarrow\ & |z_k| - 1 \geq \frac{1}{|\lambda|} \\ \Leftrightarrow\ & |z_k| \geq \frac{1}{|\lambda|} + 1 \end{aligned}$$



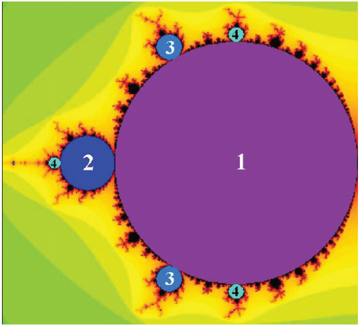
Example (First iteration $z_0 = \frac{1}{2}$)

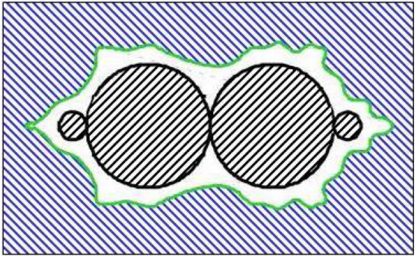
$$\begin{aligned} & \left|\frac{\lambda}{4}\right| \geq \frac{1}{|\lambda|} + 1 \\ \Leftrightarrow\ & |\lambda|^2 + 4|\lambda| - 4 \geq 0 \end{aligned}$$

Fragility in the Mandelbrot Set

What is easy

- Regional ('Global') proofs for the cyclic regions (in both z and λ).
- Proofs for the 2 period lobes are linearized z space ('global' in λ).
- Outer bounds for the set.
- The fragility of the unresolved points is easily established.
  - " White region is fragile" is a robust theorem and has a short proof.
  - Membership in white region is fragile and has complex proof.





Outline

Summary

Summary

How might this help with organized complexity and robust yet fragile?

- Long proofs indicate a fragility.
  - Either a true fragility (a useful answer) or artifact of the model (which must then be rectified).
- This example is much simpler than general dynamical systems where we cannot visualize things.
- SOS methods and tools (SOSTOOLS) give general purpose method to generate short proofs for Mandelbrot set and other dynamical systems.