RMM 26 Feb 08

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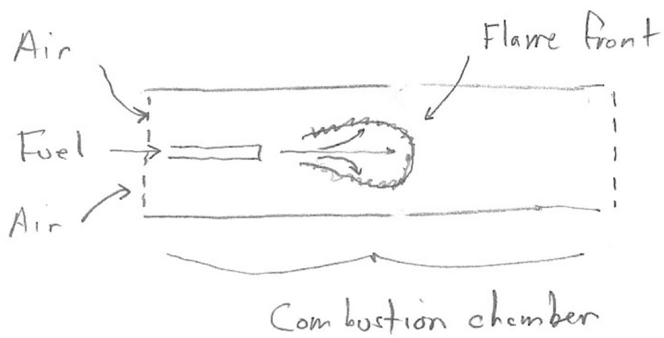
CDS 140b: Control of Limit Cycles

Motivating Example: Combustion Instabilities



Essential physics:

- Fuel + air mix at inlet
- Fuel/air mixture burns and releases heat (pressure source)
- Flame area depends on pressure

Culick model: η = acoustic mode amplitude, A_f = normalized flame are

$$\ddot{\eta} + 2\alpha\dot{\eta} + \omega_i^2 \eta = N \frac{d}{dt}\Big((1 + A_f)\Delta H_m(\dot{\eta}(t-\tau), \phi) \Big)$$

$$\dot{A}_f = b(\sigma\dot{\eta} - A_f)$$

fuel/air ration (input)

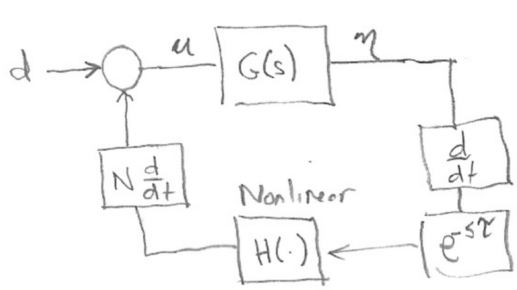
Nonlinear heat release model

time delay (nozzle → flame front)

coupling parameter (gain)

Remarks

1. Model has 3 states + time delay ⇒ infinite dimensional
2. Model is mainly linear, w/ single nonlinearity



Rough analysis: if phasing of acoustics and heat release line up → get instability leading to limit cycle.

3. Bifurcation around $(\eta, \dot{\eta}, A_f) = 0$: Homework

Bifurcation control of limit cycles

Consider a nonlinear control system

$$(*) \qquad \dot{x} = f(x, u, \mu) \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R},\ \mu \in \mathbb{R}$$

Assumptions

A1. f smooth

A2. For $u = 0$, $\exists$ nominal eq pt $x_e(\mu)$

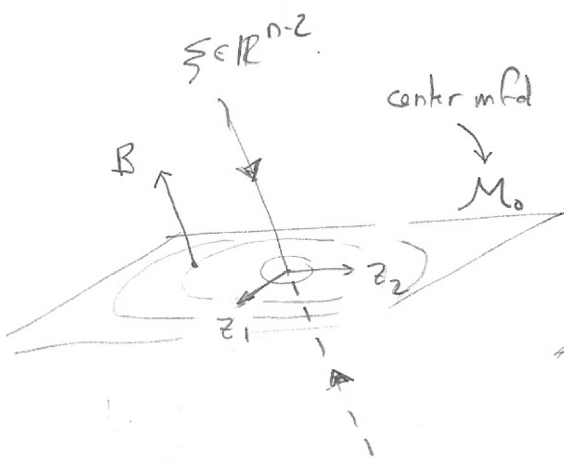
* A3. $\exists$ a pair of eigenvalue $\lambda(\mu)$, $\lambda^*(\mu)$ for the linearization $(x_e(\mu), 0)$ with $\operatorname{Im}(\lambda(\mu)) \neq 0$, $\frac{d}{d\mu} \operatorname{Re}(\lambda(\mu)) \neq 0$

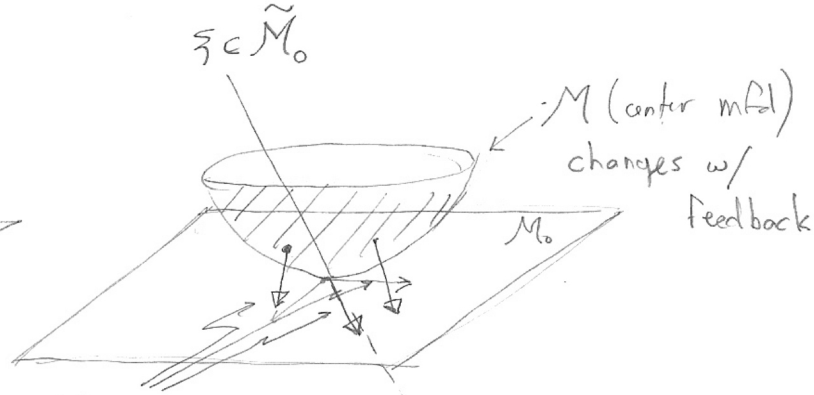
A4. The eigenspace for $\lambda(\mu)$ is not reachable for the linearization and all other eigenspaces are stable.

Q: Under what conditions can we cause Hopf bifurcation to be subcritical ($\Rightarrow$ limit cycles grow slowly, no hysteresis, etc)

A1: Redo algebraic analysis (Abed & Fu, Wong & M, Kang, etc)

A2: Slightly more geometric formulation:





$$\frac{df}{du} \sim B(z)$$

$$\frac{d}{dt}\begin{bmatrix} z \\ \xi \end{bmatrix} = A \begin{bmatrix} z \\ \xi \end{bmatrix} + Bu \qquad\qquad \dot{x} = f(x, u, \mu)$$

Geometric formulation: convert to center mfd coordinates at $\mu = \mu^* = 0$

$$A = \frac{\partial f}{\partial}(0,0) \qquad \begin{bmatrix} l \\ l^* \\ \tilde{l} \end{bmatrix} A \begin{bmatrix} r & r^* & \tilde{r} \end{bmatrix} = \begin{bmatrix} i\omega & & 0 \\ & -i\omega & \\ 0 & & A_0 \end{bmatrix}$$

$$B = \frac{\partial f}{\partial u}(0,0)$$

$$LR = I \qquad lB = 0$$

Change of coordinates: Let $z \in \mathbb{C}$ parametrize the center manifold:

$z = l x$

$z^* = l^* x$

$\tilde{x} = \tilde{l} x$

$u = \beta(z, z^*)$ [nonlinear]

$\xi = \beta(u, z, z^*)$

$\mathcal{M} = \{ x_{\mathcal{M}} \in \mathbb{R}^n : x_{\mathcal{M}} = (\mathrm{Re}(z), \mathrm{Im}(z), \beta(u, z, z^*)\}$

"Controlled" center manifold (depends on control law κ)

Geometric objects:

$$\left.\frac{df}{du}\right|_{\mathcal{M}}^{u=0} := \frac{\partial f}{\partial x}(\bar{x}, 0)\frac{\partial x_{\mathcal{M}}}{\partial u}(\bar{x}) + \frac{\partial f}{\partial u}(\bar{x}, 0)$$ Control vector evaluated on $\mathcal{M}$

$\bar{x} := (\mathrm{Re}(z), \mathrm{Im}(z), \beta(0, z, z^*))$ Point on (uncontrolled) center manifold

$\mathcal{M}_0 := \{ x \in \mathbb{R}^n \mid x = rz + r^* z^*, \ z \in \mathbb{C} \}$

$\tilde{\mathcal{M}}_0 := \{ x \in \mathbb{R}^n \mid x = \tilde{r}\xi, \ \xi \in \mathbb{R}^{n-2} \}$ ← Linear subspaces

$P_r :=$ projection to r along $\tilde{\mathcal{M}}_0 \oplus \mathrm{span}\{r^*\}$

$L_r :=$ directional derivative along r: $L_r f = \frac{\partial f}{\partial x} r$

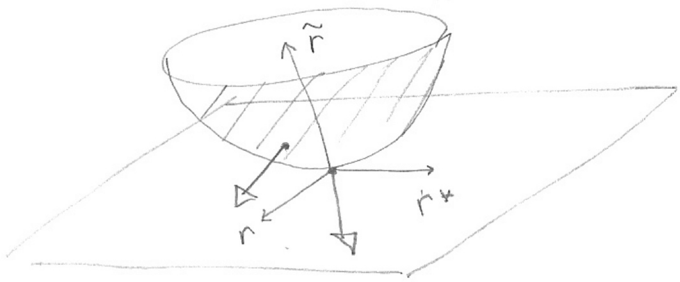
Thm (Wang, 2000) System is stabilizable at $\mu = \mu^*$ if either

1) $\mathrm{Re}\, L_r \left\langle l, \left.\frac{df}{du}\right|_{\mathcal{M}}^{u=0} \right\rangle \neq 0 \Rightarrow u = K|z|^2, \ K \in \mathbb{R}$

2) $L_{r^*} \left\langle l, \left.\frac{df}{du}\right|_{\mathcal{M}}^{u=0} \right\rangle \neq 0 \Rightarrow u = Kz^2 + K^* z^{*2}, \ K \in \mathbb{C}$

Remarks:

1. Conditions capture the ability of the input to nonlinearly affect the dynamics on the center manifold



$$\left\langle l, \frac{df}{du}\Big|_{M}^{u=0} \right\rangle = \text{Effective input vector on center mfd } (l \cdot r = 1)$$

$$L_r(\quad " \quad) = \text{nonlinear terms as I move on center mfd}$$

Use of Re, r^*, etc makes sure we get real-valued controls.

2. Pf follows approach for bifurcation of eq pts (extended to Hopf)

3. Example: Moore-Greitzer with stall amplitude and phase:

$$\dot{\Phi} = \frac{1}{l_c}\left(-\Psi + \Psi_c(\Phi) + \Psi_c'' |z|^2\right)$$

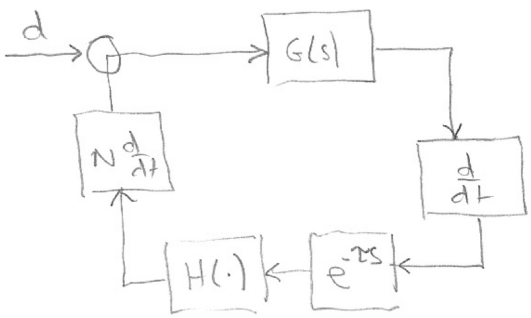
$$\dot{\Psi} = \frac{1}{4B^2 l_c}\left(\Phi - \Phi_T(\Psi, \gamma)\right)$$

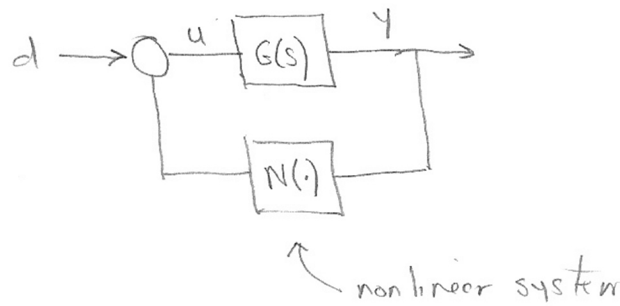
$$\dot{z} = \frac{1}{m+\mu}\left(\left(\Psi_c'(\Phi) - i\lambda\right) z + \frac{1}{2} \Psi_c''' |z|^2 z\right)$$

$$\Rightarrow \quad u = K|z|^2 = KJ$$

Loop Analysis of Nonlinear Systems

If nonlinearities enter in a simple way, can often find more constructive techniques:





Approach #1: small gain thm

Let $H: L_2(\mathbb{R}) \to L_2(\mathbb{R})$ be a linear operator taking square integrable signals to square integrable signals. Define

$$\|H\|_\infty := \sup_{\|u\|_2 \le 1} \|H(u)\|$$

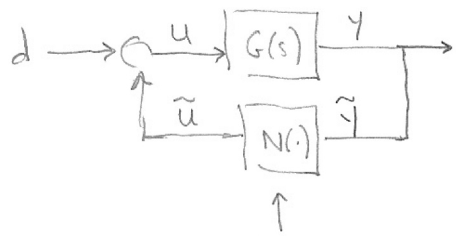
Thm (small gain) If H_1 and H_2 are connected in a feedback loop then system is stable if $\|H_1 \circ H_2\|_\infty < 1$

Remarks:

1. In frequency domain, this says $\|L(s)\|_\infty < 1 \Rightarrow$ can't get encirclements of -1 pt $\Rightarrow$ stable via Nyquist.
2. Various nonlinear extensions: circle criterion, ISS-Lyapunov, etc
3. Only useful when we can stabilize the system...

Approach #2: Harmonic balance

Idea: try to find nonzero solution of loop equations



static nonlinearity

$$\tilde{y} = \sum_{i=1}^{N} a_i e^{i\omega_i t}$$

$$\tilde{u} = \sum_{i=1}^{N} \left(M_i(a_i) e^{i(\omega_i t + \phi_i(a_i))} + \text{harmonics} \right)$$

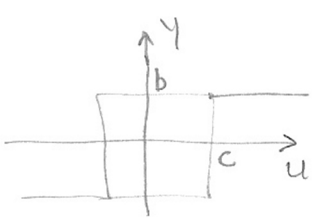
$$u = -\sum_{i=1}^{N} N(a_i) a_i e^{i\omega_i t} + \text{harmonics}$$

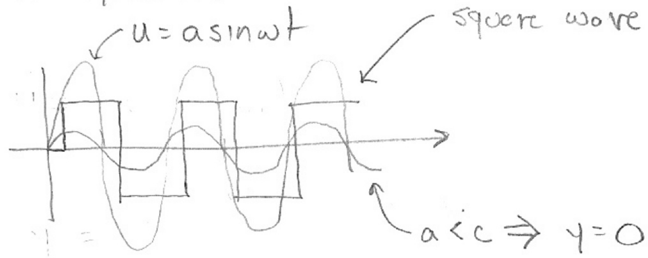
Get nonzero solution if $y = \tilde{y}$:

$$y = -\sum N(a_i, \omega_i) G(i\omega) a_i e^{i\omega_i t} + \text{harmonics} \stackrel{?}{=} \sum_{i=1}^{N} a_i e^{i\omega_i t}$$

Can use this when harmonics are small compared to primary frequency (often true for highly resonant systems since $G(i\omega)$ is small away from resonance)

Example (AM08): relay with hysteresis





Can show switching times occur at $\omega t = \sin^{-1}\left(\frac{c}{a}\right) + n\pi$

First harmonic: $y_1(t) = \left(\frac{4b}{\pi}\right) \sin(\omega t - a)$

$$N(a) = \frac{4b}{a\pi}\left(\sqrt{1 - \frac{c^2}{a^2}} - i\frac{c}{a}\right) \qquad a \geq c$$

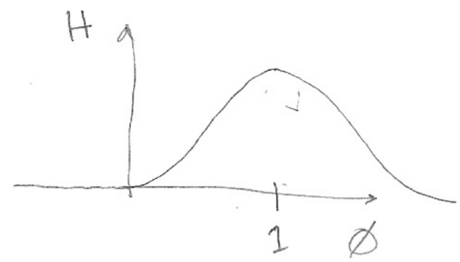
Harmonic balance applied to combustion model

$$\ddot{\eta} = -\omega_0^2 \eta - 2\alpha\dot{\eta} + N\frac{d}{dt}\left[H(\dot{\eta}(t-\tau))(1+A)\right]$$

$$\dot{A} = b(\sigma\dot{\eta} - A)$$

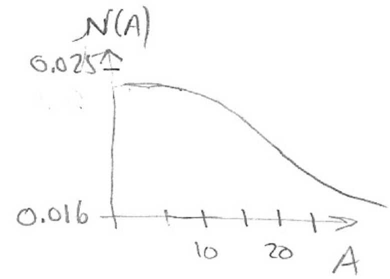
The nonlinear heat release term can be approximated by

$$H(\phi) = \begin{cases} 0 & \phi < 0 \\ N_s\left(1 - \dfrac{(\phi/k)^p}{1+(\phi/k)^p}\right) & 0 < \phi \le 1 \\ \phi e^{(1-\phi)} & \phi > 1 \end{cases}$$

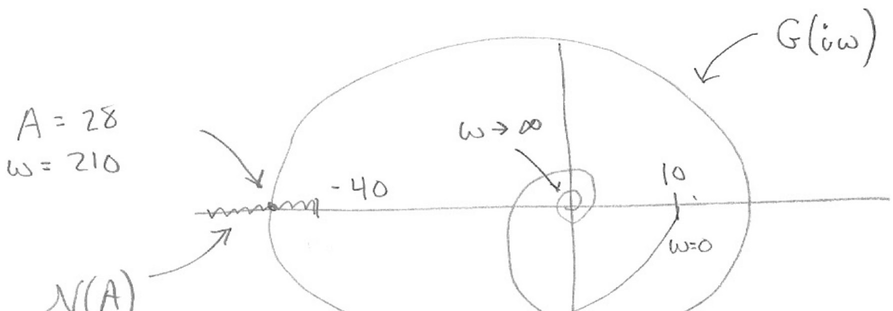


$$\phi = \frac{\bar{\phi}}{1+\dot{\eta}(t-\tau)}$$

$$\mathcal{N}(A) = \frac{M_1(A)}{A}$$ by numerical computation



$$G(s) = \frac{N s^2}{s^2 + 2\alpha s + \omega_0^2} e^{-s\tau}$$



Remarks

1. Effect of control: modify either $G(s)$ or $\mathcal{N}(A)$