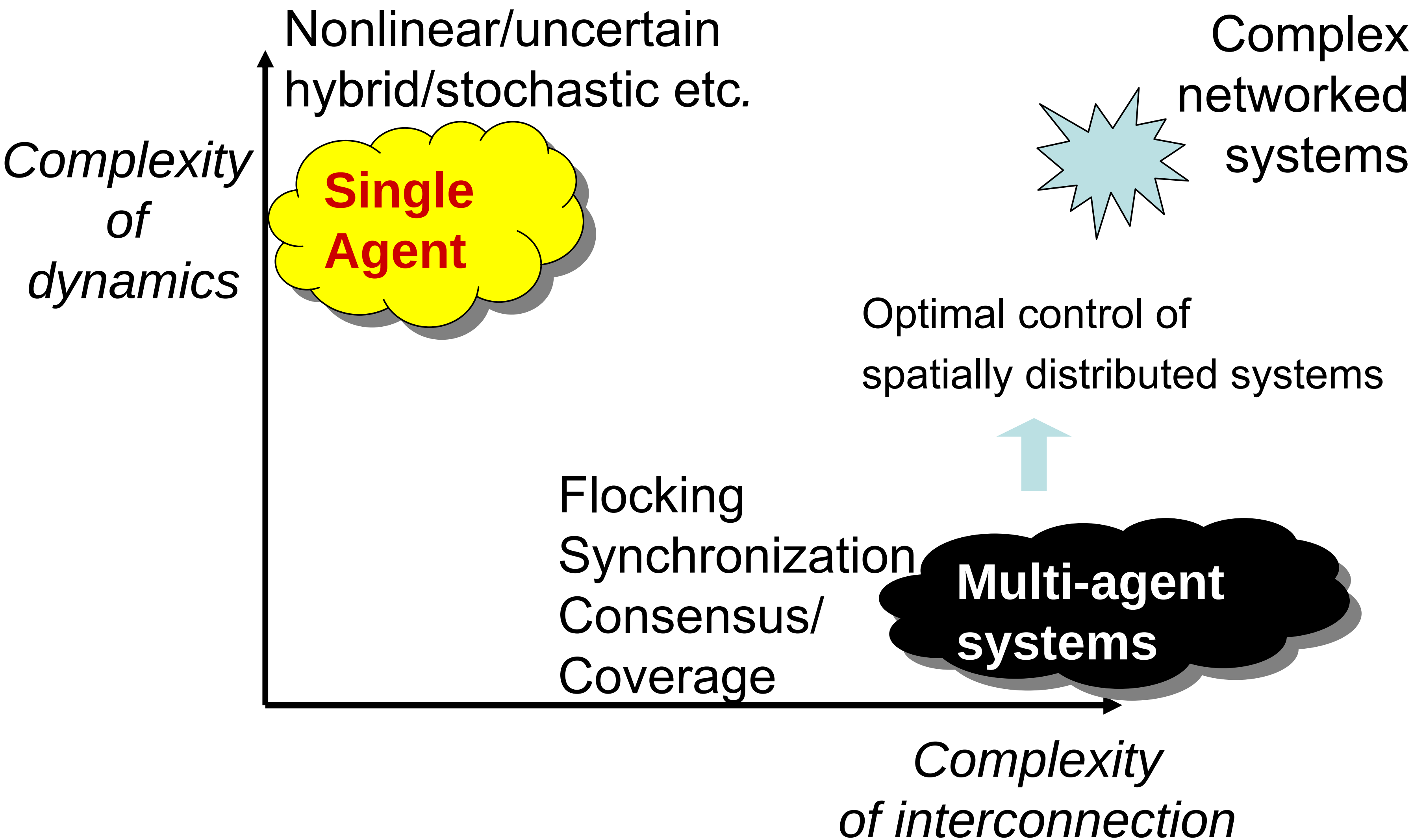
Distributed Coordination : From Flocking and Synchronization to Coverage in Sensor networks

Ali Jadbabaie

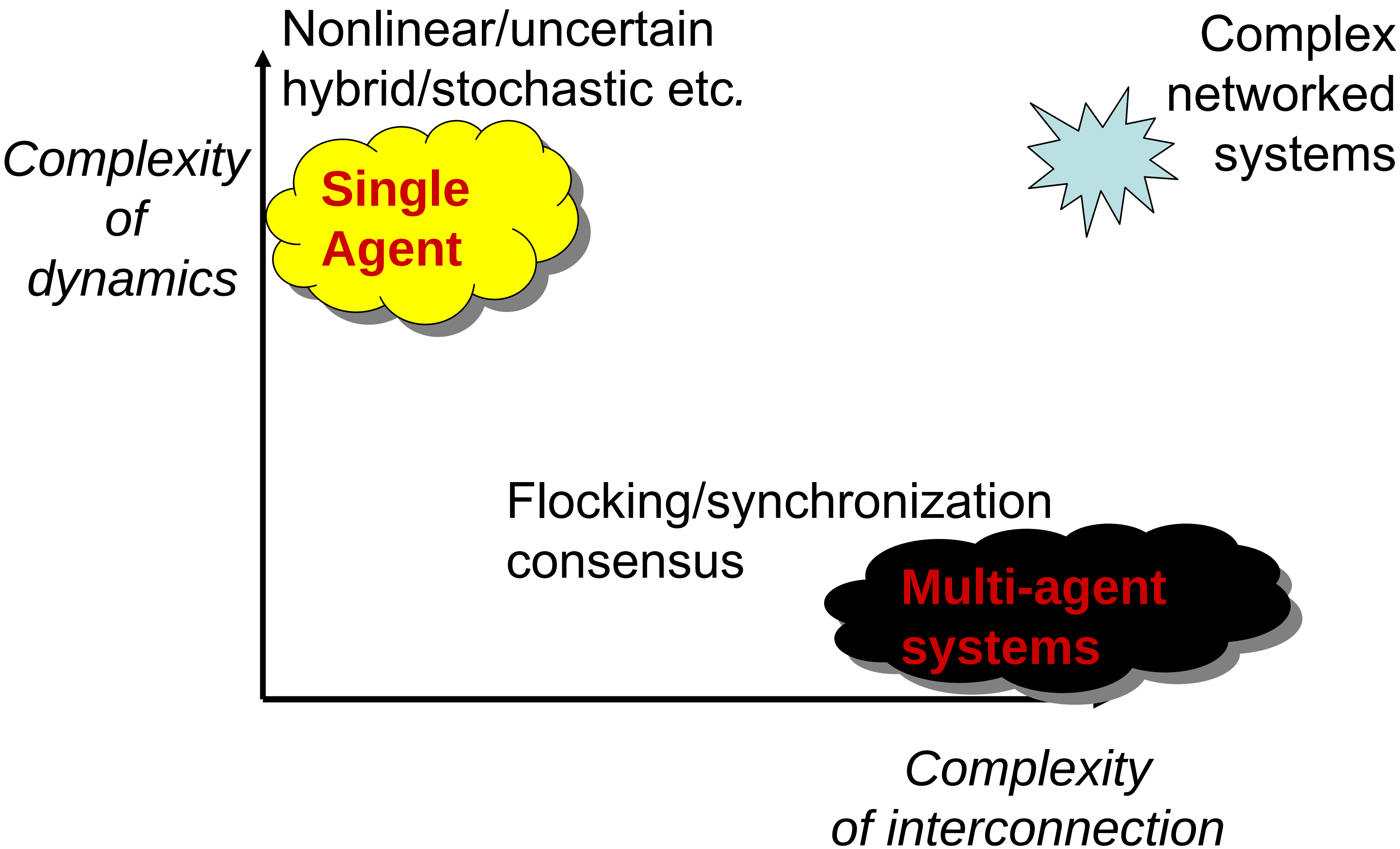
Department of Electrical and Systems Engineering
and GRASP Laboratory
University of Pennsylvania

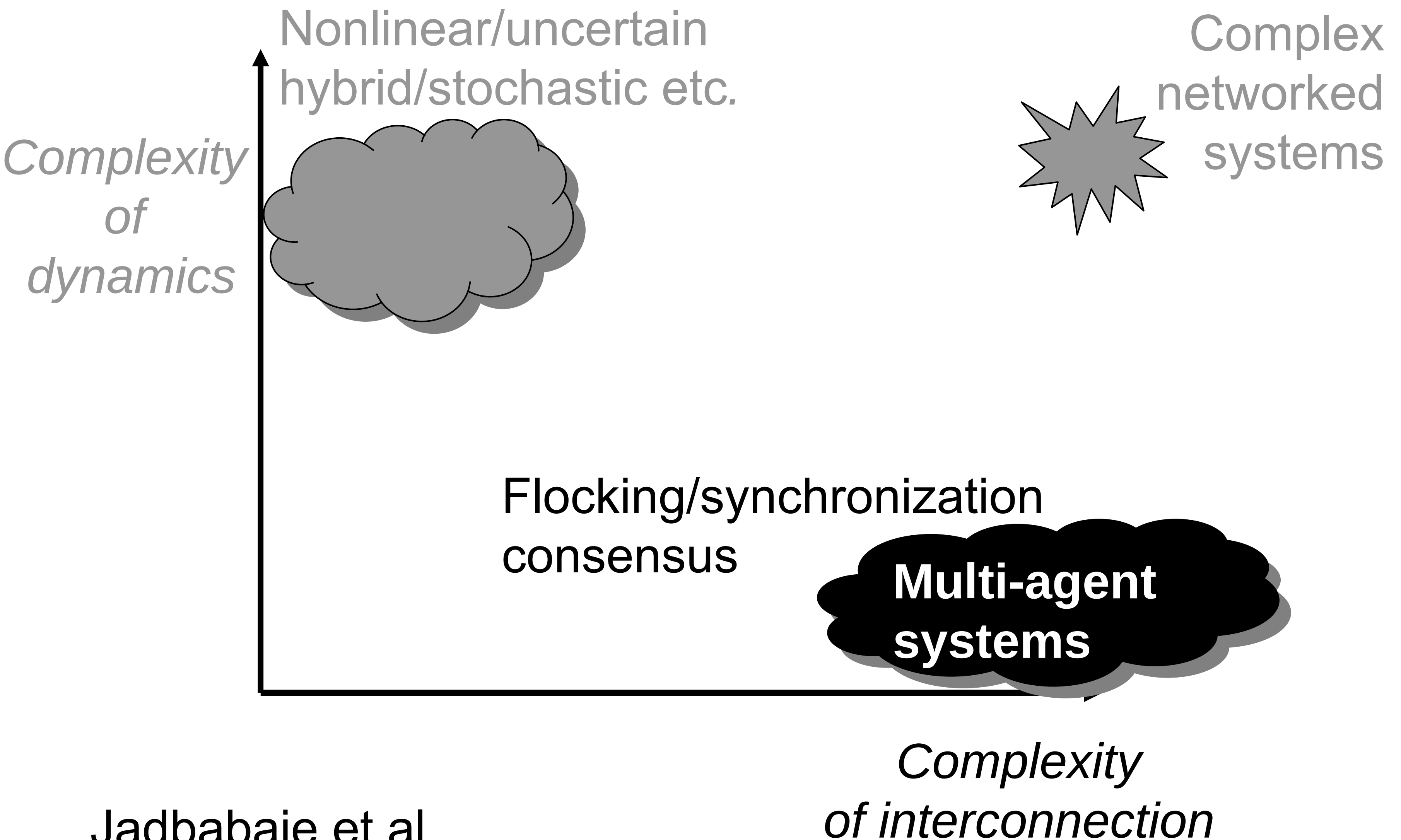
With: Nader Motee, Alireza Tahbaz Salehi, Antonis Papachristodoulou,
Abubakr Muhammad, Bert Tanner, Mauricio Barahona, Jie Lin

Networked dynamical systems



Networked dynamical systems





Jadbabaie et al

Statistical Physics and emergence of collective behavior

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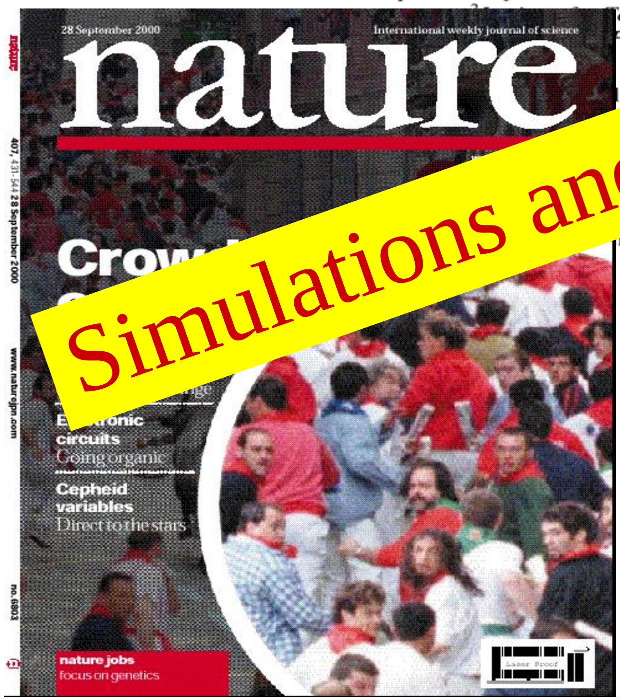
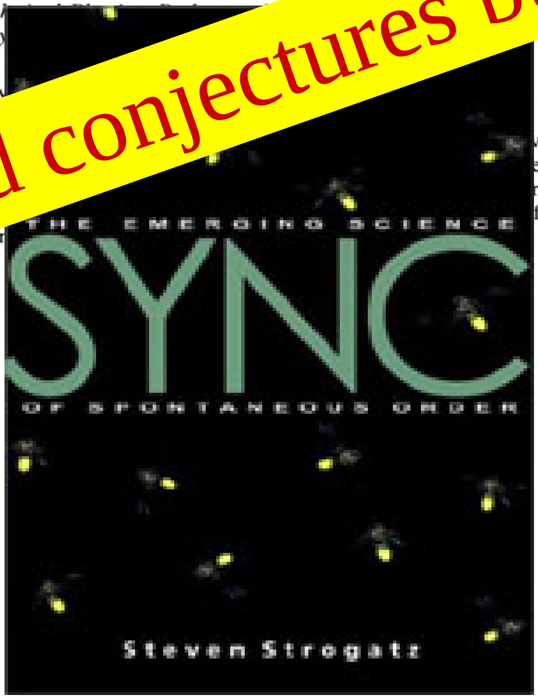
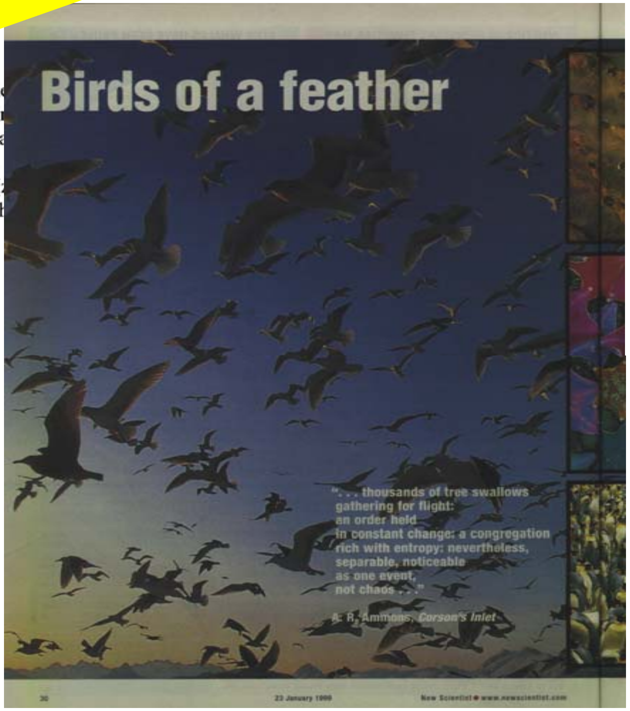
Novel Type of Phase Transition in a System of Self-Driven Par…

Tamás Vicsek,[1,2] András Czirók,[1] Eshel Ben-Jacob,[3] Inon Cohe…

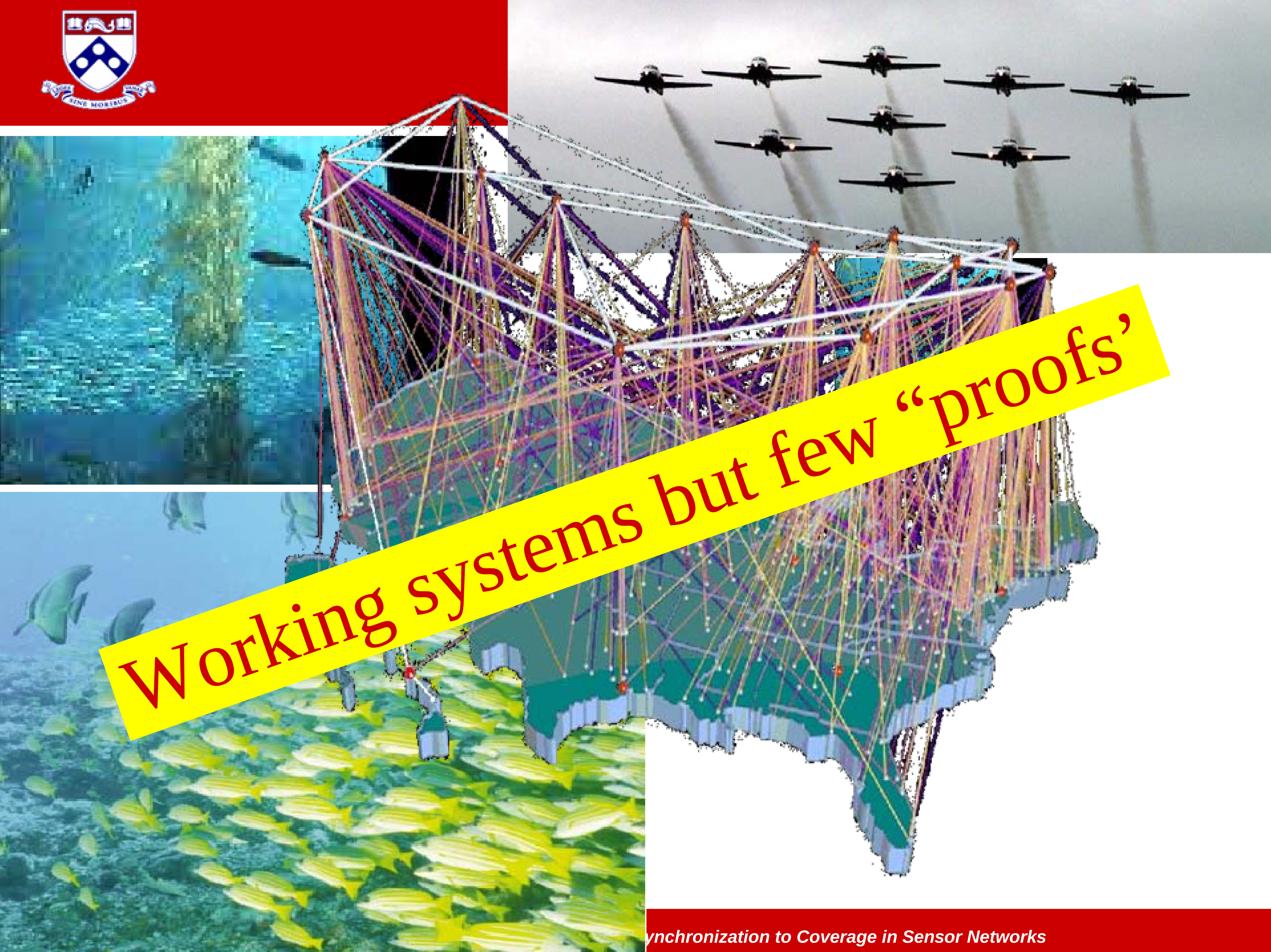
[1]*Department of Atomic Physics, Eötvös University, Budape…*



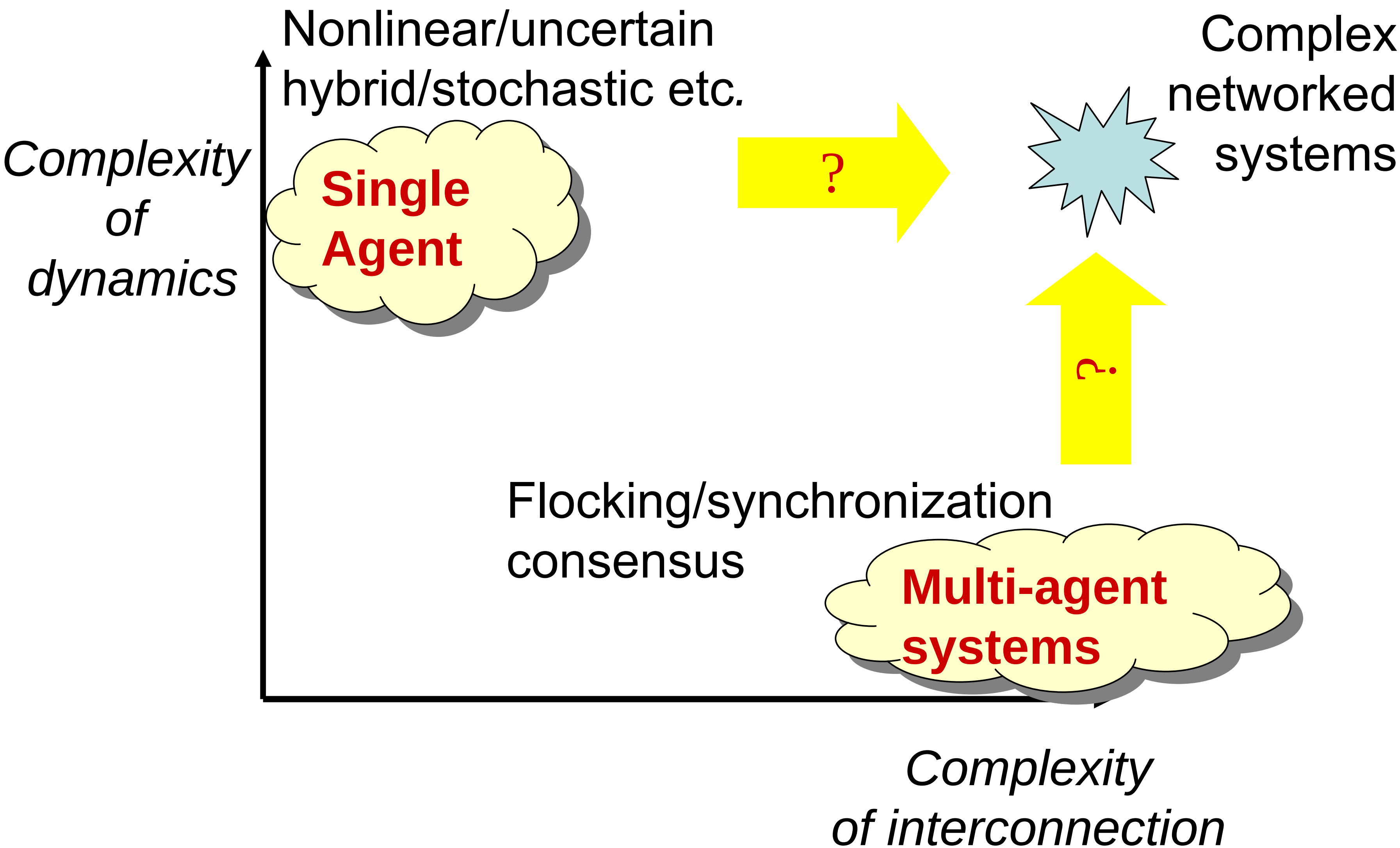




Simulations and conjectures but few "proofs'

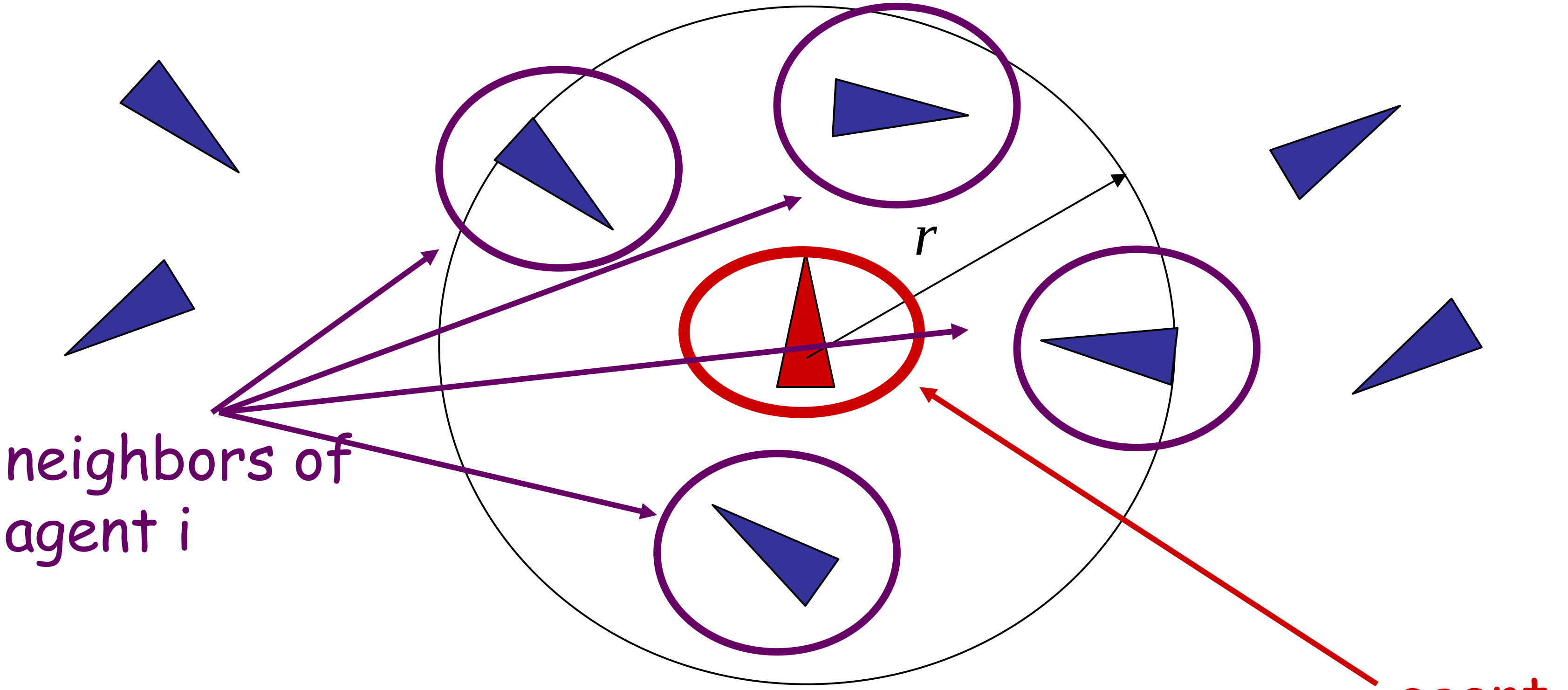

Working systems but few “proofs’

Overview


Nonlinear/uncertain
hybrid/stochastic etc.
Complex
networked
systems
Complexity
of
dynamics
Single
Agent
?
?
Flocking/synchronization
consensus
Multi-agent
systems
Complexity
of interconnection

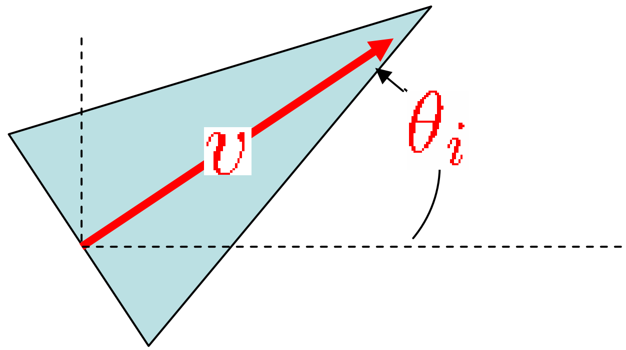
Multi-agent setting: Vicsek's kinematic model

• How can a group of moving agents collectively decide on direction, based on nearest neighbor interaction?



How does <u>global behavior</u> emerge from <u>local interactions</u>?

Distributed consensus algorithm for kinematic agents



v = speed

θ_i = heading

MAIN QUESTION : ***Under what conditions do all headings converge to the same value and agents reach a consensus on where to go?***

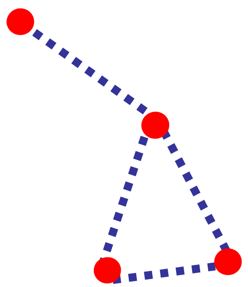
$$\theta_i(k+1) = <\theta_i(k)>_r := \operatorname{atan}\frac{\left(\sum_{j\in\mathcal{N}_i(k)} \sin\theta_j(k)\right) + \sin\theta_i(k)}{\left(\sum_{j\in\mathcal{N}_i(k)} \cos\theta_j(k)\right) + \cos\theta_i(k)}$$

For small angles

$$<\theta_i(k)>_r = \frac{1}{d_i(k)+1}\left(\sum_{j\in\mathcal{N}_i(k)} \theta_j(k) + \theta_i(k)\right)$$

Multi-agent Representations: Proximity Graphs

We use graphs to model neighboring relations



$$\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{W}\}$$

- V: A set of vertices indexed by the set of mobile agents.
- E: A set of edges the represent the neighboring relations.
- W: A set of weights over the set of edges.

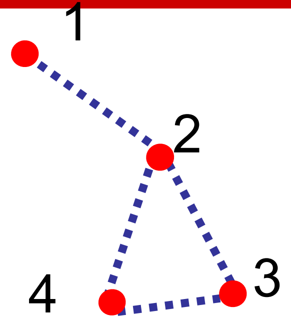
Agent i's neighborhood $\mathcal{N}_i \doteq \{j | i \sim j\}$

The neighboring relation is represented by a **fixed graph** G, or a **collection of graphs** $G_1, G_2, \ldots G_m$

The Laplacian of the graph

B is the (n x e) incidence matrix of graph G.

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$



- The graph Laplacian (n x n) encodes structural properties of the graph

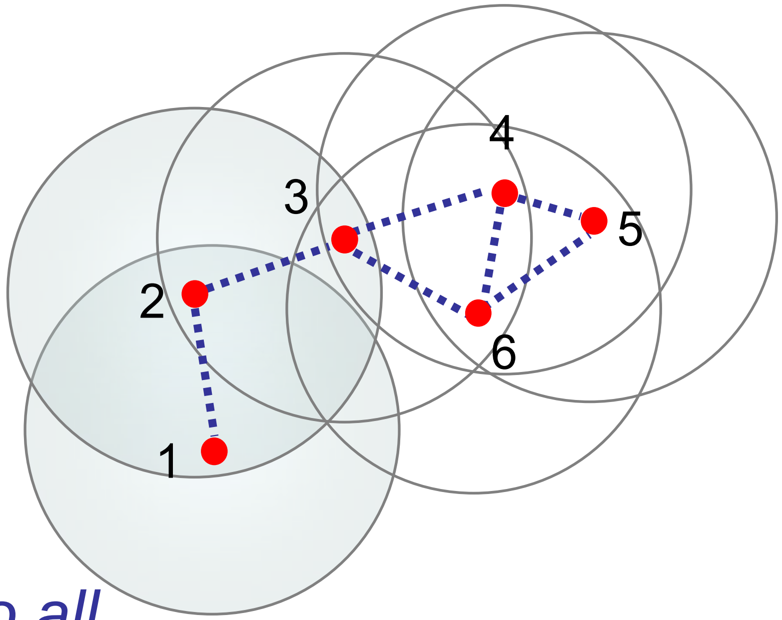
  $L = BB^T$ $\quad L_w = BWB^T$ W is diagonal

- Some properties of the Laplacian:
  - It is positive semi-definite
  - The multiplicity of the zero eigenvalue is the number of connected components
  - The kernel (for connected graph) is the span of vector of ones,

    $Lv = 0 \quad \rightarrow \quad v \in span\{\mathbf{1}\}$
  - First nonzero eigenvalue is called algebraic connectivity.
  - Its corresponding eigenvector, called the Fiedler vector. Its sign paper encodes a lot of information about "bottlenecks" and "cutpoints"

The underlying proximity graph

- We use graphs to represent neighboring relations $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$
  - vertices: $\mathcal{V} = \{1, \ldots, 6\}$
  - edges: $\mathcal{E} = \{(1,2),(2,3),(3,4),(3,6),(4,5),(4,6),(5,6)\}$



$\sigma : \{0, 1, \ldots\} \to \mathcal{P}$ switching signal ,

$\mathcal{P}$ *finite set of indices corresponding to all graphs over* n *vertices.*

A_p adjacency matrix

D_p Valence matrix

$$F_p := (D_p + I)^{-1}(A_p + I), \; p \in \mathcal{P},$$

$$\theta := \begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_n \end{bmatrix}'$$

$$\boxed{\theta(k+1) = F_{\sigma(k)}\theta(k)}$$

Necessary and sufficient condition for convergence of products of stochastic matrices

Theorem (Wolfowitz '63, Daubechies & Lagarias '92, '01)
All infinite products of stochastic matrices chosen from a finite set $\Sigma = \{F_1, \cdots, F_m\}$ converge to a rank-one matrix $\mathbf{1c}$ for some row vector $\mathbf{c}$, if and only if:

All finite products $F_{i_1} F_{i_2} \cdots F_{i_k} \; \forall k > 0$ of all lengths are ergodic matrices, where $F_{i_j} \in \Sigma,\ j = \{1, \cdots, m\}$

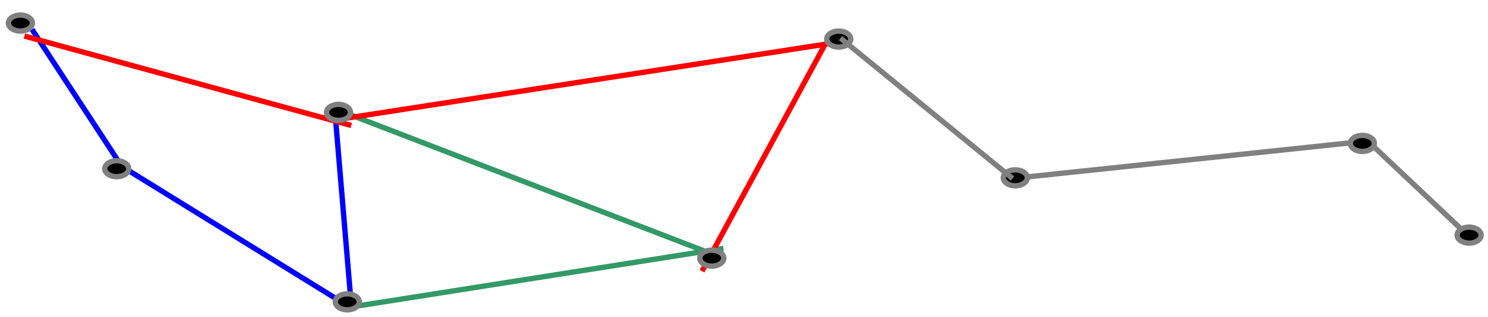
Finite product ergodicity ⟺ Ergodicity of Σ

- Complexity: decidable but PSPACE-Complete (Hernek'95).
- The necessary and sufficient condition does not provide an effective computation scheme. Need to exploit the problem structure.
- Products of ergodic matrices is not necessarily ergodic.

Conditions for reaching consensus

Theorem (Tsitsiklis'84, Jadbabaie et al. 2003): *If there is a sequence of bounded, non-overlapping time intervals T_k, such that over any interval of length T_k, the network of agents is "jointly connected ", then all agents will reach consensus on their velocity vectors.*

This happens to be both necessary and sufficient for exponential coordination, boundedness of intervals not required for asymptotic coordination. (see Moreau '04)



Consensus in random networks

$$F_p := (D_p + I)^{-1}(A_p + I), \; p \in \mathcal{P},$$

$$\theta := \begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_n \end{bmatrix}' \qquad \boxed{\theta(k+1) = F_{\sigma(k)}\theta(k)}$$

$\sigma : \{0, 1, \ldots\} \to \mathcal{P}$ is a random switching signal

Theorem : Assuming graphs are randomly chosen and independent, reaching consensus is a trivial event, i.e., either it happens almost surely or almost never, i.e., it satisfies the Kolmogorov 0-1 law.

Theorem: necessary and sufficient condition for almost sure convergence is $\lambda_2(E(F)) < 1$,i.e., the average matrix needs to be ergodic.

Consensus in Continuous time

$$\dot{\theta}(t) = -(D_{\sigma(t)}+I)^{-1}(D_{\sigma(t)}-A_{\sigma(t)})\theta(t) := -(D_{\sigma(t)}+I)^{-1}L_{\sigma(t)}\theta(t)$$

- As before, $\sigma(t)$ is a piecewise constant switching signal
- The model is now a hybrid or switching dynamical system
- Need to assume a dwell time on each graph to avoid complications
- The result is virtually the same, as exponentials of Laplacians are stochastic matrices

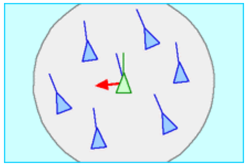
lemma If $\{\mathbb{G}_{p_1}, \mathbb{G}_{p_2}, \ldots, \mathbb{G}_{p_m}\}$ is a jointly connected collection of graphs with Laplacians $L_{p_1}, L_{p_2}, \ldots, L_{p_m}$, then

$$\bigcap_{i=1}^{m} \text{kernel } L_{p_i} = \text{span } \{\mathbf{1}\}. \qquad (1)$$

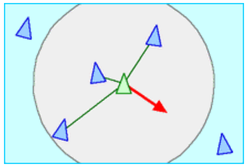
Later on go from graphs to *simplicial complexes and use this to verify coverage*

Flocking, artificial life, and computer graphics

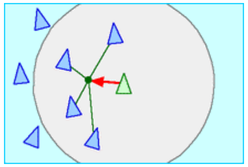
- Reynolds [Reynolds 87] named the autonomous systems that behave like members of animal groups boids (bird + oids)
- He developed a descriptive model for flocking behavior based on the combined action of alignment and cohesion-separation forces



alignment: steer towards the average heading of flockmates



separation: steer to avoid crowding flockmates



cohesion: steer towards the average position of flockmates

Distributed coordination with dynamic models:

- Double integrator model

$$\dot{r}_i = v_i$$

$$\dot{v}_i = u_i = a_i + \alpha_i$$

- Neighbors of i distance dependent:

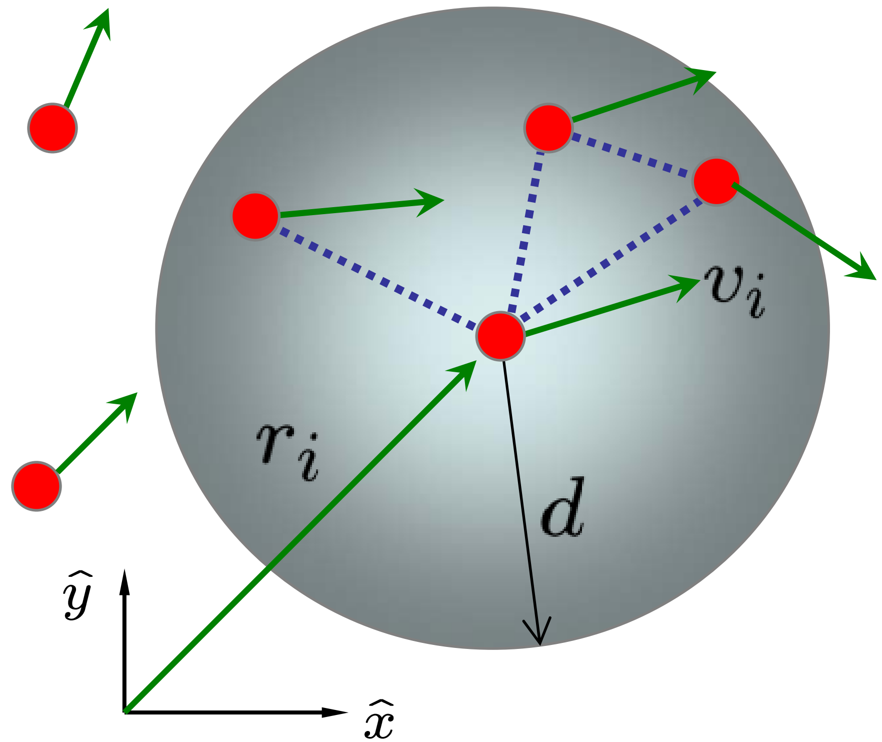
$$\mathcal{N}_i \subset \{1, \dots, N\}$$

- Cohesion/Separation

$$a_i = -\nabla V_i = -\nabla \sum_{j \in \mathcal{N}_i} V_{ij}(\|r_{ij}\|)$$

- Alignment

$$\alpha_i = -k \sum_{j \in \mathcal{N}_i} (v_i - v_j)$$



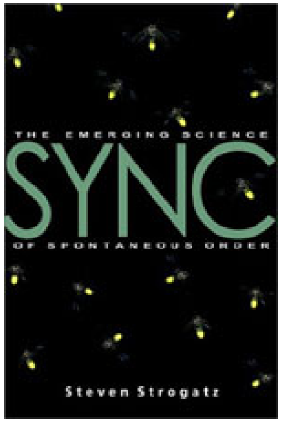
For dynamic models, Proximity graph Connectivity implies emergence of Collective motion (Tanner, Jadbabaie, Pappas)

Synchronization of coupled oscillators

- Kuramoto model with long-range interaction

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i)$$

- Toy model for *pacemaker cells in the heart and nervous system, collective synchronization of pancreatic beta cells, synchronously flashing fire flies, gait generation for bipedal robots (Klavins and Koditschek'02).*
- Benchmark problem in physics
- Not very well understood over arbitrary networks



Kuramoto model over arbitrary graphs

Only neighboring oscillators contribute to the sum

$$\dot{\theta}_i = \omega_i - \frac{K}{N} \sum_{j \in \mathcal{N}_i} \sin(\theta_i - \theta_j) \qquad \dot{\theta} = \omega - \frac{K}{N} B \sin(B^T \theta)$$

B is the incidence matrix

$$B \sin(B^T \theta) = B W(\theta) B^T \theta$$

$$W(\theta) = diag([\sin(B^T \theta)_1 / (B^T \theta)_1 \cdots \sin(B^T \theta)_e / (B^T \theta)_e])$$

Theorem: For arbitrary connected networks, connectivity implies local stability of the synchronized state. Rate of convergence determined by "algebraic connectivity", the first non-zero eigenvalue of the graph Laplacian.

Properties of the model

- When ω=0, $\theta = \theta_{ss}\mathbf{1}$ is an asymptotically stable fixed point.

$$r(t)^2 = \frac{1}{N^2}\left|\sum_{i=1}^{N} \cos(\theta_i) + j\sin(\theta_i)\right|^2 = (e^{j\theta})^*(NI - L)e^{j\theta}$$

$$= \frac{N^2 - 2e + 2\mathbf{1}^T \cos(B^T\theta)}{N^2}$$

r(t) is the order parameter, or measure of synchrony

- $U := N^2(1 - r^2) = [e^{j\theta}]^* L [e^{j\theta}]$ is a Lyapunov function, measuring asynchrony.

$$\dot{U}(\theta) = \nabla U(\theta)\dot{\theta} = -\frac{2}{KN}\dot{\theta}^T\dot{\theta} \leq 0.$$

$\lambda_2(L)$ determines the speed of synchronization

A Special Case

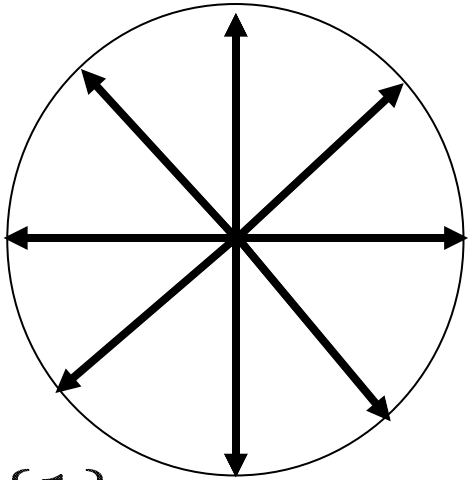
$$\theta \in Null\{L_w\} = span\{\mathbf{1}\}$$

For |θi|<π/2 for a connected graph, all trajectories will converge to S

$$S = \{\theta_i \mid \theta_i = \theta_j, \forall i \neq j\}$$

Therefore, all velocity vectors will synchronize.

But, this stability result is not global. In the case of the ring topology $\theta \in span\{\mathbf{1}\}$ is not the only equilibrium. This is due to the fact that B and B^T have the same null space! $\theta_i - \theta_j = \frac{2\pi}{N}$ is also stable:



Fixed points: $\theta \in span\{\mathbf{1}\}, \quad B^T\theta \in span\{\mathbf{1}\}$

Kuramoto model w/ non identical oscillators

- When the frequencies are non zero, there is no fixed point for small values of coupling.

- Theorem: *Bounds on the critical value of the coupling can be determined by maximum deviation of frequencies from the mean, and algebraic connectivity of the graph.*

$$K \geq \frac{||w-\bar{\omega}\mathbf{1}||_2}{\lambda_2(L)}$$

$$K_{ave} \geq \frac{\sqrt{N}\sigma}{\lambda_2(L)}$$

$$\sqrt{1-\frac{\lambda_{\max}(L)}{N}} \leq r \leq \sqrt{1-\frac{w^T L^{\dagger} w}{K^2}} \leq \sqrt{1-\frac{||w-\bar{\omega}\mathbf{1}||_2^2}{K^2\lambda_2(L)}}$$

- When ω is random, $r \leq \sqrt{1-\frac{Tr(L)\sigma^2}{K^2}}$

Kuramoto model w/ non identical oscillators

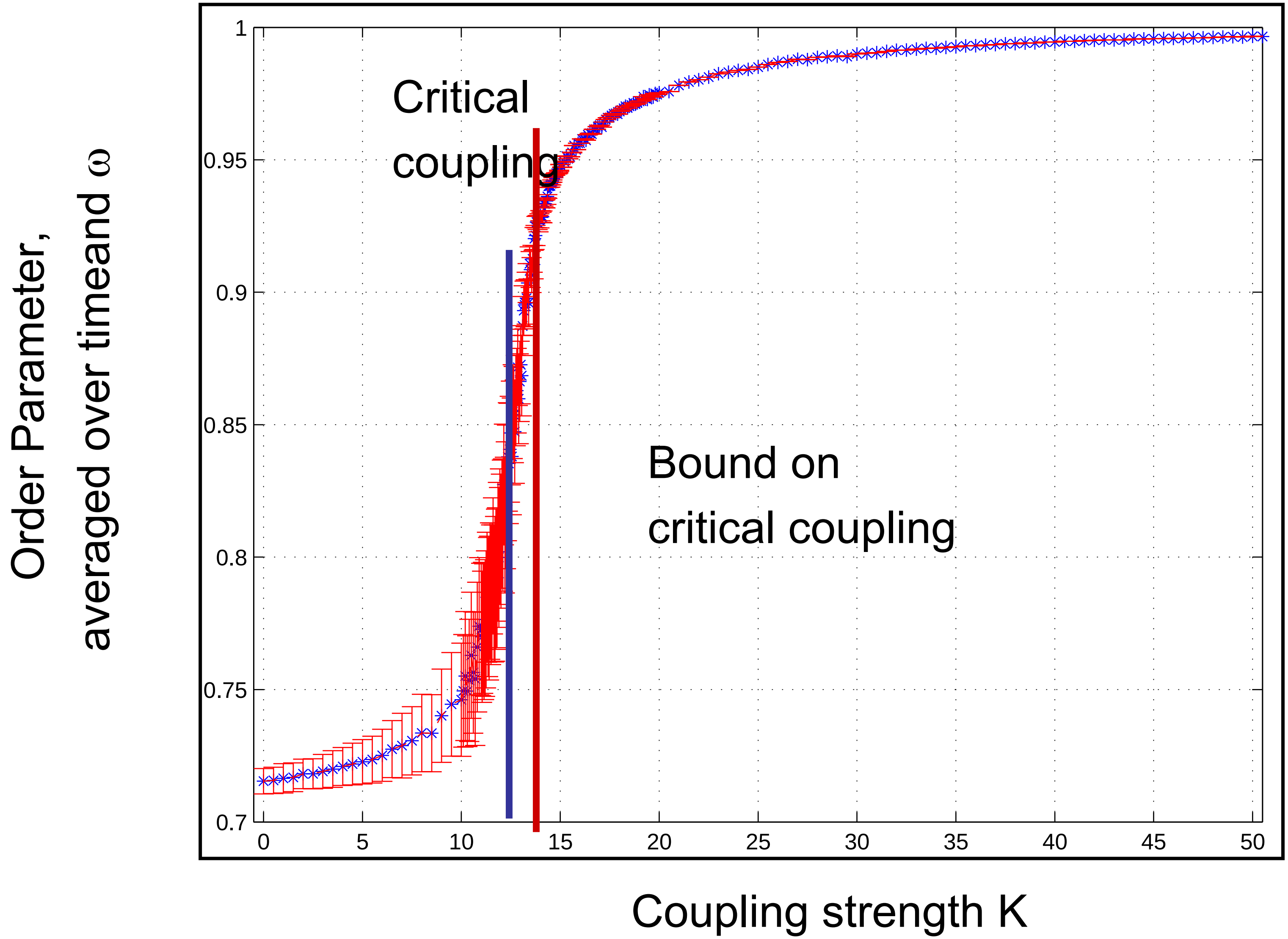
- When the frequencies are non zero, there is no fixed point for small values of coupling.
- There is no partial synchronization for fixed values of initial frequencies for small K.
- Theorem: *Bounds on the critical value of the coupling can be determined by maximum deviation of frequencies from the mean, and algebraic connectivity of the graph.*

$$K \geq \frac{||w-\bar{\omega}\mathbf{1}||_2}{\lambda_2(L)}$$

$$\sqrt{1-\frac{\lambda_{\max}(L)}{N}} \leq r \leq \sqrt{1-\frac{w^T L^\dagger w}{K^2}} \leq \sqrt{1-\frac{||w-\bar{\omega}\mathbf{1}||_2^2}{K^2\lambda_2(L)}}$$

- When ω is random, $r \leq \sqrt{1-\frac{Tr(L)\sigma^2}{K^2}}$

Average order parameter vs. K for N= 100,e= 2443



Dual decomposition and nonlinear network flow

- Want to globally minimize 1-r^2 over the whole network
- Let $z = \sin(B^T\ \theta)$

$$\min_{z_j} \sum_{j=1}^{e} (1 - \sqrt{1 - z_j^2}) = e - \mathbf{1_e}^{\mathbf{T}} \cos(\mathbf{B}^{\mathbf{T}}\theta)$$

Sum of pair-wise potentials

Subject to: $Bz = N\dfrac{\omega}{K} := s$

Supply at each node

Lagrangian $$L(z,\nu) = \sum_{j=1}^{e} 1 - \sqrt{1 - z_j^2} - (\nu^T B)_j z_j + \sum_{i=1}^{N} \nu_i s_i$$

$$g(\nu) = \inf_z L(z,\nu) \qquad \frac{z_j}{\sqrt{1 - z_j^{*2}}} = \Delta\nu_j = \tan(B^T\theta)_j \quad \Delta\nu_j$$

$$g(\Delta\nu) = \sum_{j=1}^{e} 1 - \frac{1}{\sqrt{1 + \Delta\nu_j^2}} + \sum_{1=1}^{N} \nu_i s_i$$

Kuramoto model is the Subgradient algorithm for solving the dual

$$\nu_i(k+1) = \nu_i(k) - \sum_{l \in \mathcal{N}_i} \frac{\nu_i - \nu_l}{\sqrt{1 + (\nu_i - \nu_l)^2}} + s_i$$

Subgradient algorithm

Shor 87, Tsitsiklis '86

$$\sin(\theta_i - \theta_l)$$

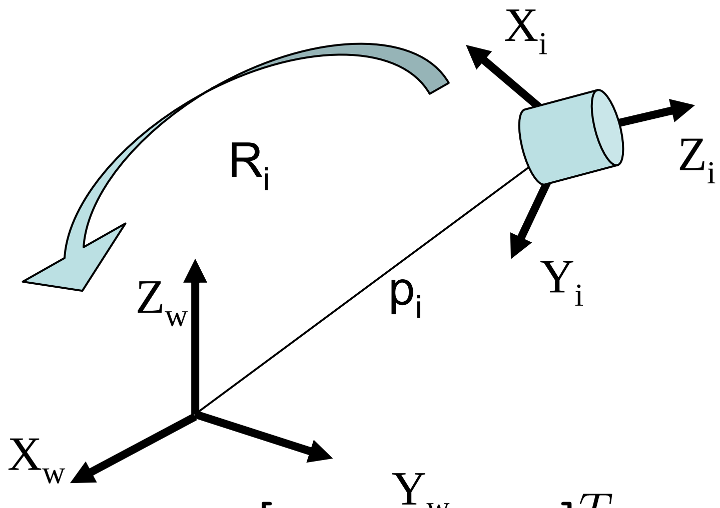
From synchronization to distributed alignment

- Velocity vector v_i of agent i is a unit vector along the z-axis of the body frame.

$$v_i = R_i e_3 \qquad R_i = [R_{ix} \ R_{iy} \ R_{iz}]$$

- Full Kinematics equation:

$$\begin{aligned} \dot{p}_i &= v_i \\ \dot{R}_i &= R_i \widehat{\omega}_i \end{aligned} \qquad \widehat{\omega}_i = \begin{bmatrix} 0 & -\omega_{iz} & \omega_{iy} \\ \omega_{iz} & 0 & -\omega_{ix} \\ -\omega_{iy} & \omega_{ix} & 0 \end{bmatrix}$$

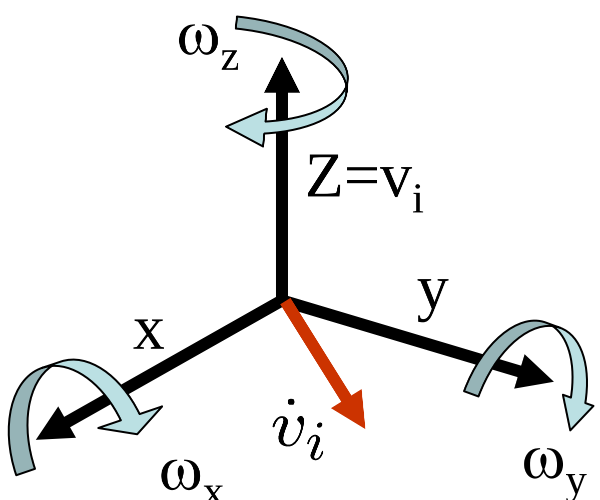


$$\omega_i = [\omega_{ix}, \omega_{iy}, \omega_{iz}]^T$$

- ω_i is the **body angular velocity**.

- The **reduced kinematics** becomes:

$$\begin{aligned} \dot{p}_i &= v_i \\ \dot{v}_i &= -\omega_{ix} R_{iy} + \omega_{iy} R_{ix} \end{aligned}$$



Distributed velocity alignment in 3D

Theorem: Consider the system of N kinematic agents

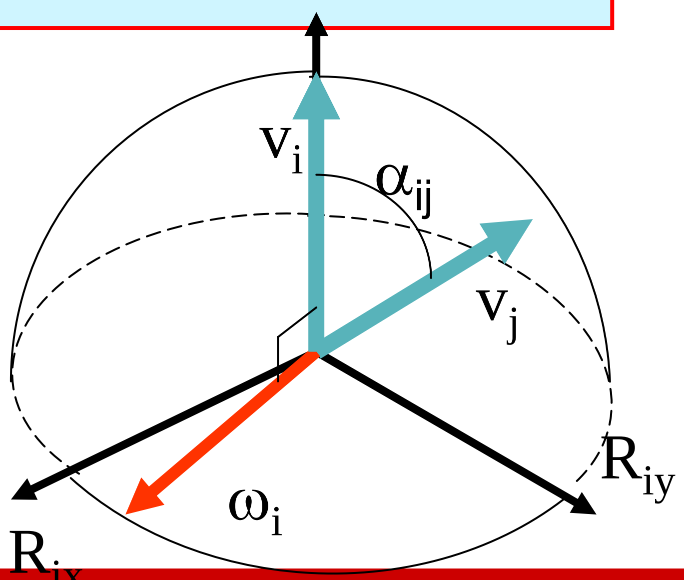
$$\dot{v}_i = -\omega_{ix} R_{iy} + \omega_{iy} R_{ix}$$

If the proximity graph of the agents is **connected over time**, and all initial velocity vectors are in a hemisphere, applying the control law

$$\omega_i = \sum_{j \in \mathcal{N}_i} v_i \times v_j$$

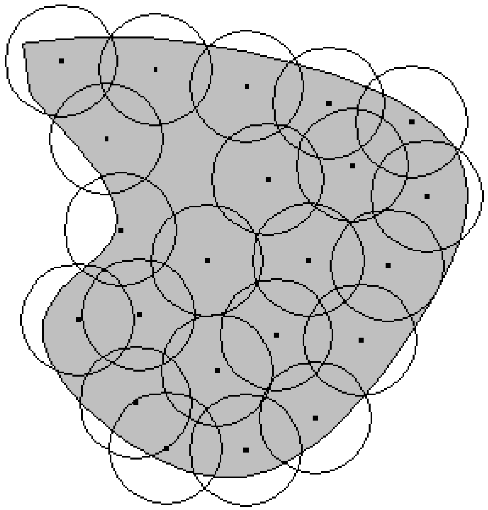
will result in asymptotic velocity alignment

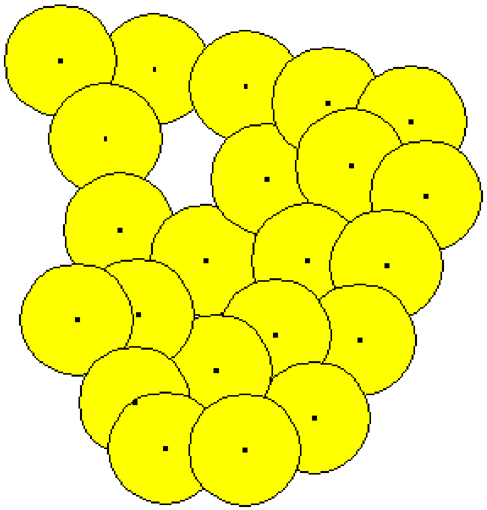
- Proof:
  - all trajectories converge to the equilibria given by $\omega_i = 0$.
  - A hemisphere is positively **invariant** under our control law.
  - The **consensus set** is the equilibrium set in the hemisphere.
  - Note that application of LaSalle to switched graph case is "tricky"



Beyond Graphs in Networked Systems

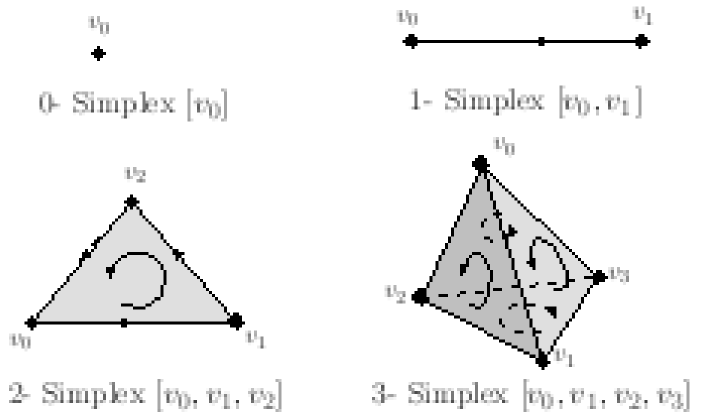
- Spectral graph theory helps us quantify properties of networked systems
- For certain problems, e.g. coverage, makes sense to go beyond graphs and pair-wise interactions
- Example: Given a set of sensor nodes in a given domain (possibly bounded by a fence), is every point of the domain under surveillance by at least one node?





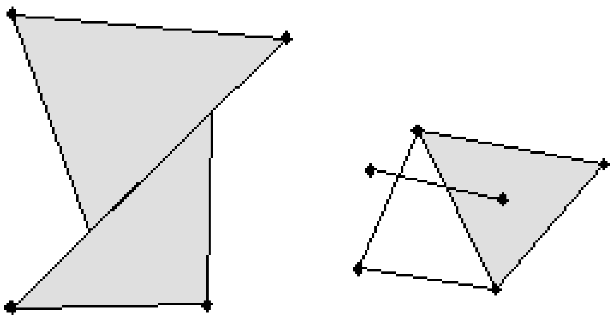
From Graphs to Simplicial Complexes

- Simplicial Complex: A finite collection of simplices
- Simplex: Given V, an unordered non-repeating subset
- k-simplex: The number of points is k+1
- Faces: All (k-1)-simplices in the k-simplex
- Orientation



From Graphs to Simplicial Complexes

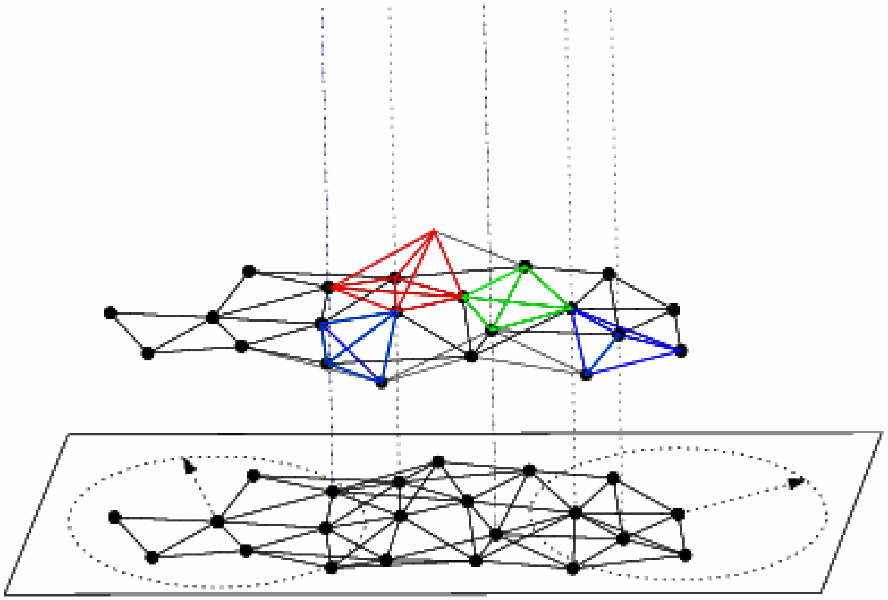
- Simplicial complex: made up of simplices of several dimensions
- Properties
  - Whenever a simplex lies in the collection then so does each of its faces
  - Whenever two simplices intersect, they do so in a common face.
- Valid Examples
  - Graphs
  - Triangulations



Invalid examples

Rips-Vietoris Simplicial Complex

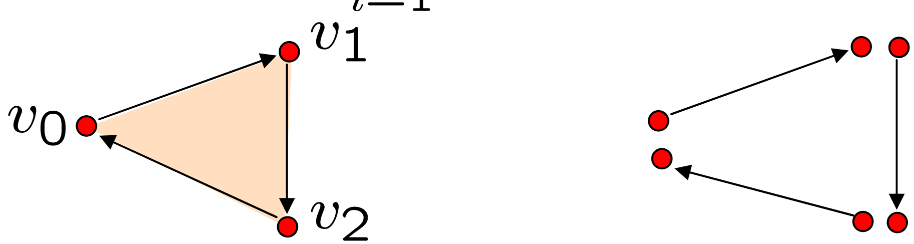
- 0-simplices : Nodes
- 1-simplices : Edges
- 2-simplices: A triangle in the connectivity graph ~ 2-simplex (Fill in with a face)
- K-simplices: a complete subgraph on k+1 vertices
- k-simplex in the Rips complex ~ (k+1) points within communication range of each other



Boundary Maps

- **Let $C_k(X)$ be the vector space whose basis is the set of oriented k-simplices of X**
- **The** boundary map $\partial_k : C_k \rightarrow C_{k-1}$ **is the linear transformation**

$$\partial_k[v_0, \ldots, v_k] = \sum_{i=1}^{k} (-1)^i [v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k]$$



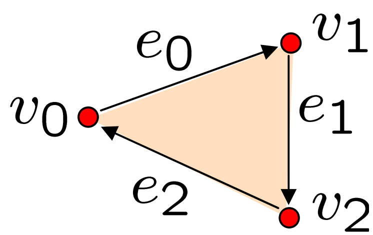
$$\partial_2[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

- k-cycles: $Z_k(X) = \ker(\partial_k : C_k \rightarrow C_{k-1})$
- k-boundaries: $B_k(X) = \mathrm{im}(\partial_{k+1} : C_{k+1} \rightarrow C_k)$
- **Note:** $\partial_k \circ \partial_{k+1} = 0$

Homology groups : $H_k(X) = Z_k(X) / B_k(X)$

Combinatorial k-Laplacians

- **Since X is finite we can represent the boundary maps in matrix form**



$$"\partial_1" = B = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

incidence matrix

- **Moreover, we can get the** adjoint

$$\partial_k^\star : C_{k-1}(X) \to C_k(X)$$

- **[Eckmann 1945]** **The** Combinatorial k-Laplacian $\Delta_k : C_k(X) \to C_k(X)$ **is given by**

$$\Delta_k = \partial_{k+1}\partial_{k+1}^\star + \partial_k^\star \partial_k$$

- **Note:**

$$\Delta_0 = BB^T = L$$

k-Laplacian at the Simplex Level

- Adjacency of a simplex to other simplices
  - Upper adjacency if they share a higher simplex (e.g. 2 nodes connected by an edge) $\sigma_i \frown \sigma_j$
  - Lower adjacency if they share a common lower simplex (e.g. two edges share a node) $\sigma_i \smile \sigma_j$
- 'Local' formula with orientations $\epsilon_{ij} \in \{-1, 1\}$

$$\mathcal{L}_k(\sigma_i) = (\deg_u(\sigma_i) + k + 1)\sigma_i + \sum_{\sigma_i \smile \sigma_j} \epsilon_{ij}\sigma_j - \sum_{\sigma_i \frown \sigma_m} \epsilon_{im}\sigma_m$$

Homology classes from k-Laplacians

- **The** harmonic k-cycles **are given by (Hodge theorem)**

$$\mathcal{H}_k(X) = \{c \in C_k(X) \mid \Delta_k c = 0\}$$

- **We now have a decomposition into orthogonal subspaces**

$$C_k(X) = \mathcal{H}_k(X) \oplus \mathrm{im}(\partial_{k+1}) \oplus \mathrm{im}(\partial_k^\star)$$

- **The Laplacian operator is** invariant **on each subspace and** positive definite **on** $\mathrm{im}(\partial_{k+1}),\ \mathrm{im}(\partial_k^\star)$ (i.e. on $\mathcal{H}_k(X)^\perp$)

- **Unique harmonic cycle for each homology class**

- **Kernel of the Laplacian ~ homology classes**

Laplacian Flows

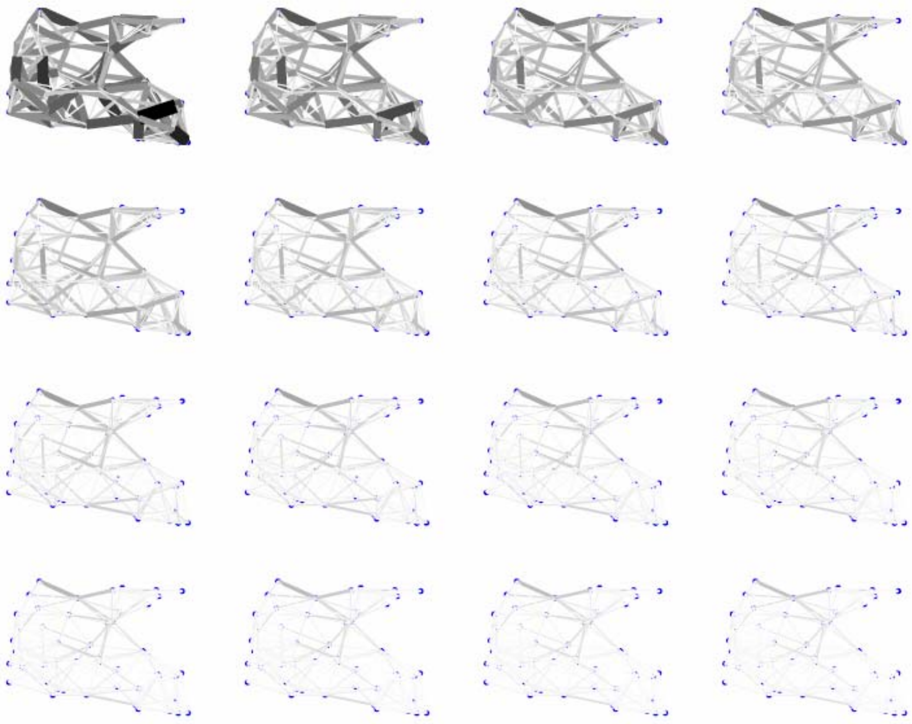
- Laplacian flows : a semi-stable dynamical system

$$\frac{\partial \omega}{\partial t} = -\Delta_k \omega, \qquad k \geq 0,$$

(Recall heat equation for k = 0)

- [Muhammad-Egerstedt MTNS'06]

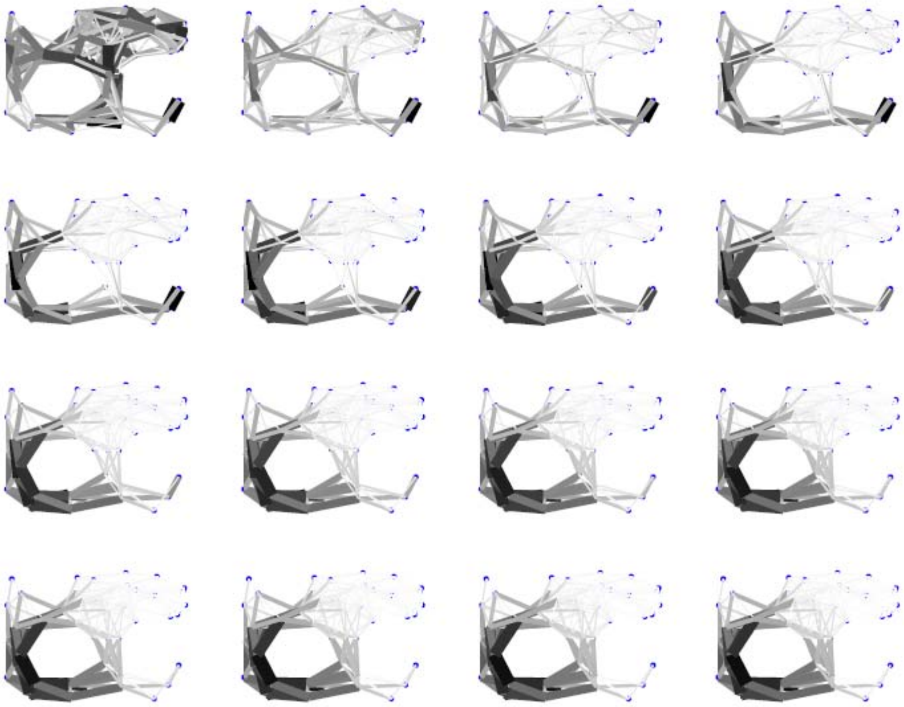
System is asymptotically stable if and only if rank($H_k(X)$) = 0.



- A method to detect 'no holes' locally

Laplacian Flows (contd.)

- System converges to the unique harmonic cycle if rank($H_k(X)$) = 1.
- A method to detect 'proximity to hole' locally when single hole

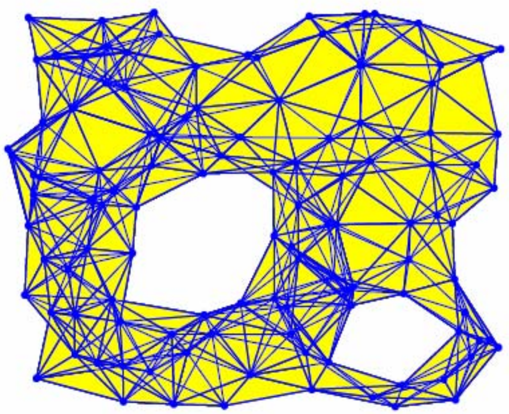


- When rank($H_k(X)$) > 1 :
  System converges to the span of harmonic homology cycles

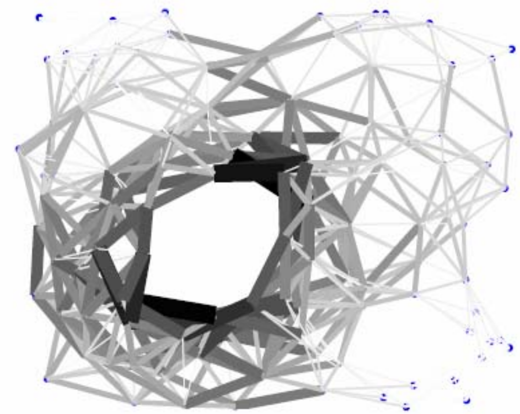
Work in Progress ….

- Distinguish between multiple homology classes by decentralized eigenvector decomposition of k-Laplacian (Kempe's algorithm)
- Quantify 'proximity to holes'
- Quantify fragilities in network : near-harmonic cycles (Fiedler like characterization such as cutpoints for holes?)
- Switching k-Laplacians and 'wandering holes'
- A "spectral theory" for simplicial complexes?

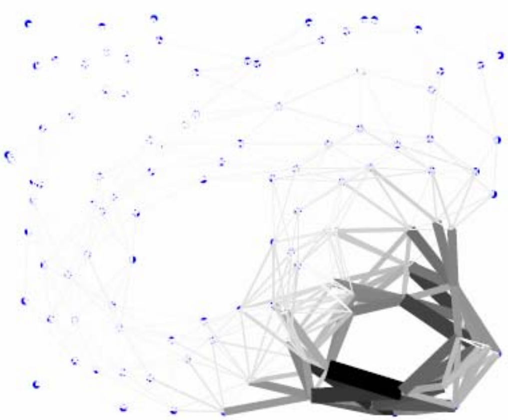
Example, eigenvectors of L_1



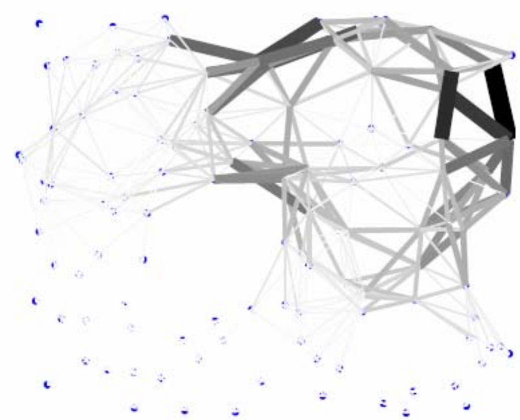
Network



1st homology class

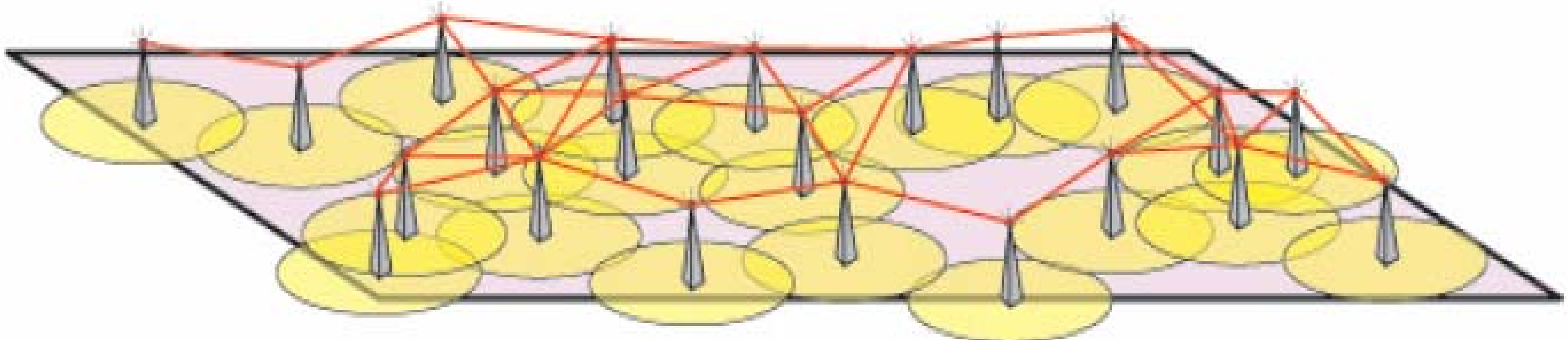


2nd homology class



‘Fiedler-like’- eigenvector

Ongoing work: detection of wandering holes in coverage



V. De Silva and R Ghrist, "Homological Sensor Networks"

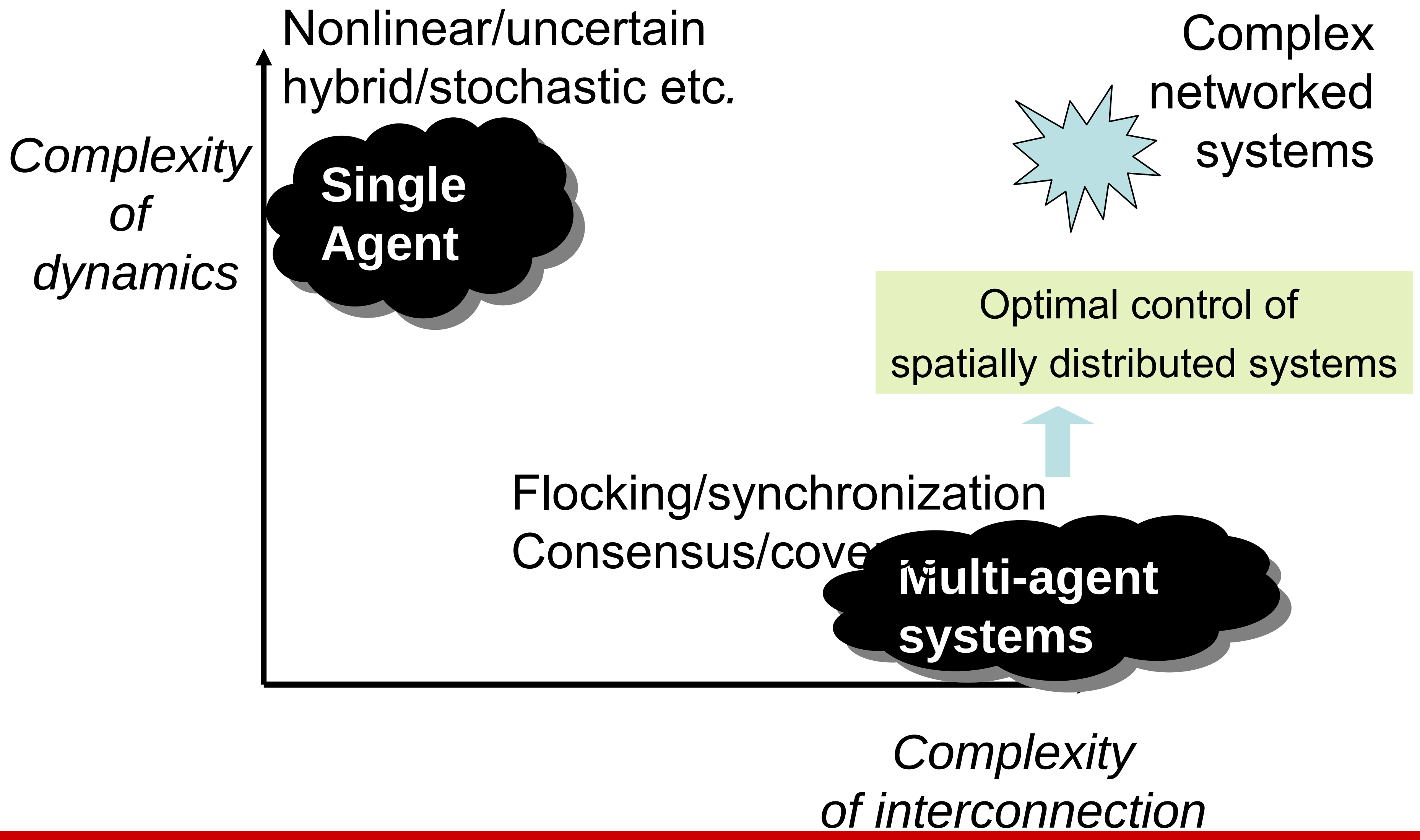
- Given a set of sensors with a disk footprint, add:
  - an edge when 2 sensors overlap. A face when 3 sensors overlap
- Construct the 1st Laplacian L_1

$$\dot{x} = -L_1^{\sigma(t)} x(t) \qquad \sigma : \{0, 1, \ldots\} \to \mathcal{P}$$

Rips complex is "jointly connected over time" ⟺ intersection of kernels of Laplacians is zero ⟺ no wandering hole in coverage ⟺ The dynamical system (which is distributed) converges to zero

Instead of Spectral Graph theory look at spectral theory of simplicial complexes

Networked dynamical systems



Results on Spatially invariant systems and distributed control

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 47, NO. 7, JULY 2002 1091

Distributed Control of Spatially Invariant Systems

Bassam Bamieh, *Member, IEEE*, Fernando Paganini, *Member, IEEE*, and Munther A. Dahleh, *Fellow, IEEE*

1478 IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 48, NO. 9, SEPTEMBER 2003

Distributed Control Design for Spatially Interconnected Systems

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 49, NO. 12, DECEMBER 2004 2113

Distributed Control of Heterogeneous Systems

Geir E. Dullerud and Raffaello D'Andrea

- Mostly over highly symmetric graphs w/ identical dynamics
- Infinite Horizon Quadratic Cost
- No constraints on inputs and states

1502 IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 49, NO. 9, SEPTEMBER 2004

Distributed Control Design for Systems Interconnected Over an Arbitrary Graph

Cédric Langbort, Ramu Sharat Chandra, and Raffaello D'Andrea, *Senior Member, IEEE*

On the Ill-Posedness of Certain Vehicular Platoon Control Problems

Mihailo R. Jovanović, *Member, IEEE*, and Bassam Bamieh, *Senior Member, IEEE*

Abstract—We revisit the vehicular platoon control problems formulated by Levine and Athans and Melzer and Kuo. We show that ... consider the infinite platoon case as an insightful limit which can be treated analytically. We argue that the infinite platoons

1446 IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 49, NO. 9, SEPTEMBER 2004

Distributed Control of Systems Over Discrete Groups

Benjamin Recht, *Student Member, IEEE*, and Raffaello D'Andrea, *Senior Member, IEEE*

Abstract—This paper discusses distributed controller design and analysis for distributed systems with arbitrary discrete symmetry groups. We show how recent results for designing controllers for spatially interconnected systems, based on semidefinite ... would fall into this category. However, there are many spatially invariant configurations such as those arising from crystalline structures which have noncommutative symmetry groups. An

274 IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 51, NO. 2, FEBRUARY 2006

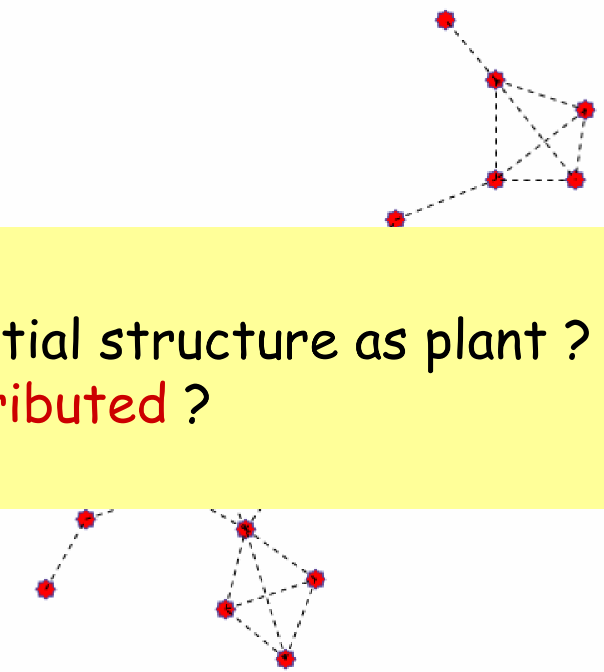
A Characterization of Convex Problems in Decentralized Control*

Michael Rotkowitz and Sanjay Lall

Structure of optimal control for spatially distributed systems: spatially invariant case

- Model of each subsystem:

$$\text{Eq.(1)} \quad \begin{bmatrix} x_k(t+1) \\ y_k(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_k(t) \\ u_k(t) \end{bmatrix}$$



Does the optimal control policy have the same spatial structure as plant ?
In other words, is it spatially distributed ?

- Finite Horizon Optimal Control problem:

$$\min_{u^N} J(x(0), u^N) \quad \longleftarrow \text{ Finite Horizon Quadratic Cost}$$

$$\begin{aligned} \text{s.t. } & \text{Eq.(1)} && \text{for } 0 \le t \le N \\ & u_{min}^k \le u_k(t) \le u_{max}^k && \text{for } 0 \le t \le N_c \\ & y_{min}^k \le y_k(t) \le y_{max}^k && \text{for } 0 \le t \le N_c \\ & k \in G \end{aligned}$$

Fourier Analysis on Lattice:

1D Lattice:

$$\begin{matrix} & \cdots & x_{-2} & x_{-1} & x_0 & x_1 & x_2 & \cdots & \\ \cdots & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \cdots \end{matrix}$$

Signals in the spatial domain: $x = (\cdots, x_{-1}, x_0, x_1, \cdots)$

Fourier transform: $\hat{x} = \cdots + x_{-1} e^{i\omega} + x_0 + x_1 e^{-i\omega} + \cdots$

For simplicity, replace $z = e^{i\omega}$

Fourier transform: $\hat{x} = \sum_{k \in G} x_k z^{-k}$

G : spatial domain

Translation Invariant Operators

- Definition:.
  - Translation Operator: $\mathbf{T}(\cdots, | x_k, x_{k+1}, \cdots) = (\cdots, | x_{k+1}, x_{k+2}, \cdots)$
  - Q is translation invariant operator if $\mathbf{T}Q = Q\mathbf{T}$

Consider translation invariant operators of this form

$$Q(\mathbf{T}) = \sum_{k \in G} Q_k \mathbf{T}^k$$

agent k is coupled to its neighbors through cost function J

$$J = \cdots + \overbrace{x_k^* Q_{-1} x_{k-1} + x_k^* Q_0 x_k + x_k^* Q_1 x_{k+1}} + \cdots$$

$$= \langle x, Q(\mathbf{T}) x \rangle$$

in which $\quad Q(\mathbf{T}) = Q_{-1} \mathbf{T}^{-1} + Q_0 + Q_1 \mathbf{T}^1$

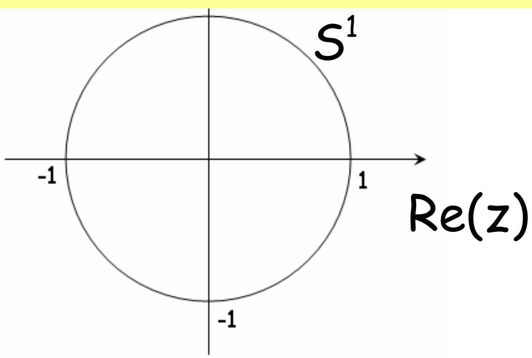
Decay Property of Translation Invariant Operators

$$Q(T) = \sum_{k \in G} Q_k T^k \qquad \hat{Q}(z) = \sum_{k \in G} Q_k z^k = \frac{1}{d(z)} N(z)$$

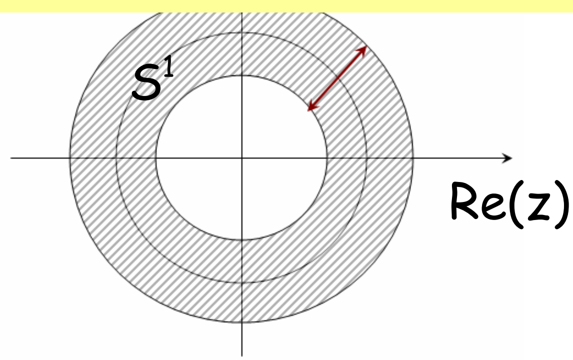
Fact 1: Analytic continuity implies decay in spatial domain.

- Analytic continuity

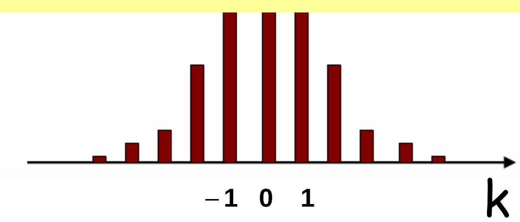
Fact 2: The decay rate depends on the distance of the closest pole to the unit circle; the further, the faster.



No pole on S^1

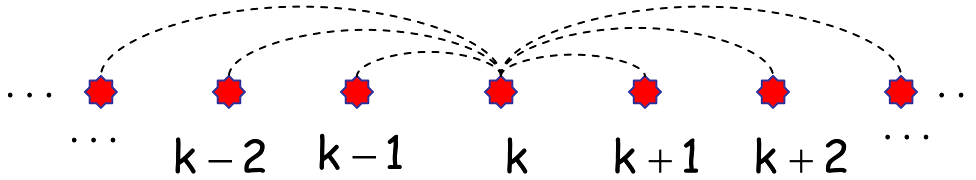


No pole in an annulus around S^1



Coefficients decay in spatial domain

Back to our problem



- Model of each subsystem:

$$\text{Eq.(1)} \quad \begin{bmatrix} x_k(t+1) \\ y_k(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_k(t) \\ u_k(t) \end{bmatrix}$$

- Notation:

$$x(t) = (\cdots, x_k(t), x_{k+1}(t), \cdots)$$

$$u(t) = (\cdots, u_k(t), u_{k+1}(t), \cdots)$$

$$\mathbf{u}^N = (\cdots, \mathbf{u}_k^N, \mathbf{u}_{k+1}^N, \cdots)$$

$$\mathbf{u}_k^N = (u_k(0), u_k(1), \cdots, u_k(N-1))^*$$

Spatial Locality of Centralized RHC

- Finite Horizon Quadratic Cost:

$$J(x(0),\mathbf{u}^N) = \langle x(N), P(\mathbf{T})x(N)\rangle + \sum_{t=0}^{N-1} \langle x(t), Q(\mathbf{T})x(t)\rangle + \langle u(t), R(\mathbf{T})u(t)\rangle$$

- $P(\mathbf{T})$ can be obtained from a parameterized family of DAREs:

$$A^* \hat{P}(z) A - \hat{P}(z) - A^* \hat{P}(z) B (\hat{R}(z) + B^* \hat{P}(z) B)^{-1} B^* \hat{P}(z) A + \hat{Q}(z) = 0$$

for all $z \in S^1$.

- $P(\mathbf{T})$ is spatially decaying:

$$P(\mathbf{T}) = \sum_{k \in G} P_k \mathbf{T}^k \longrightarrow \| P_k \| \leq c e^{-\beta |k|} \quad \text{for some } c, \beta > 0$$

Spatial Locality of the Optimal Solution

Theorem: Given the initial condition x(0), the optimal solutions are :

(1) Affine maps of x(0), i.e., $u_i^N = \sum_{j \in G} K_{ij}\, x_j(0) + c_i$

(2) Spatially distributed, i.e., $\left\| K_{ij} \right\|_2 \leq \alpha e^{-\beta |i-j|}$

for some $\alpha, \beta > 0$.

Generalization to Arbitrary Graphs

Analytic continuity → Exponential decay in spatial domain

Q_{ki} : coupling between agent k and i

Multiply by ζ where $1 \leq \zeta < b$

$$Q_{ki} \longrightarrow \tilde{Q}_{ki} = Q_{ki} \, \zeta^{dis(k,i)}$$

$$Q = \begin{bmatrix} \ddots & & \\ & Q_{ki} & \\ & & \ddots \end{bmatrix} \qquad \tilde{Q} = \begin{bmatrix} \ddots & & \\ & \tilde{Q}_{ki} & \\ & & \ddots \end{bmatrix}$$

Suppose that Q is bounded.

If $\tilde{Q}$ is bounded, then we say that Q is exponentially spatially decaying.

Note: SD stands for Spatially Decaying

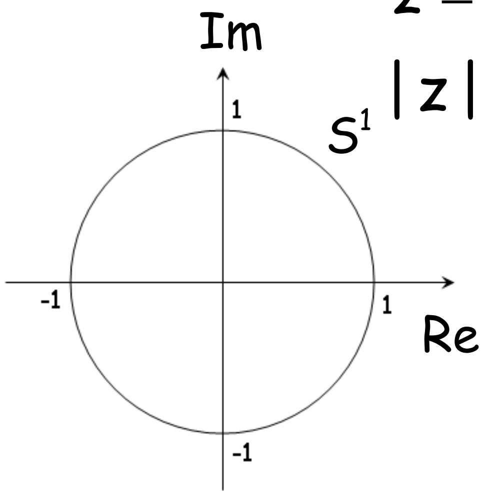
Extending analytic continuity

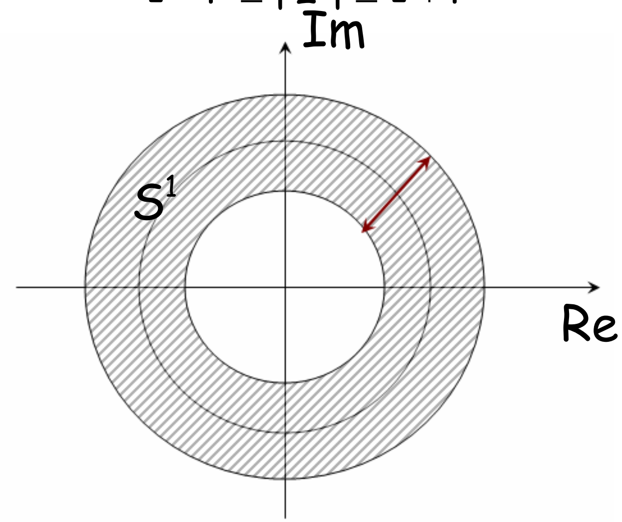
Multiply by ζ $\longrightarrow$ $\tilde{z} = \zeta e^{-i\omega}$

$z = e^{-i\omega}$ where $1 - r \leq \zeta \leq 1 + r$

$|z| = 1$

$1 - r \leq |\tilde{z}| \leq 1 + r$





No pole on S^1 $\longrightarrow$ (Analytic continuity) $\longrightarrow$ No pole in an annulus

If $\sum_{k \in G} |Q_k|_2 \, \zeta^{|k|} < \infty$ $\longrightarrow$ $\hat{Q}(\tilde{z}) = \sum_{k \in G} Q_k \tilde{z}^k$ is bounded on the annulus.

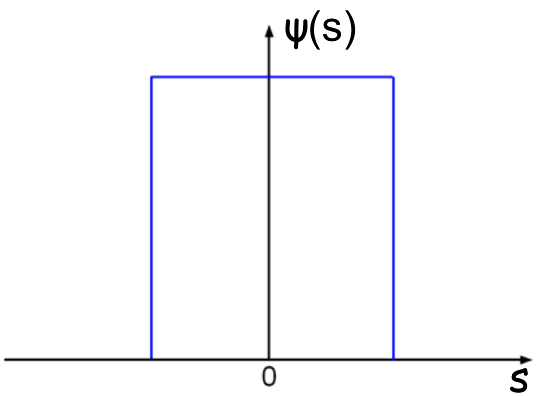
$\longrightarrow$ Q is spatially decaying.

Systems with Arbitrary Couplings over Arbitrary Graphs

Multiply by ψ(dis(k,i))

$$Q_{ki} \longrightarrow \tilde{Q}_{ki} = Q_{ki}\ \psi(dis(k,i))$$

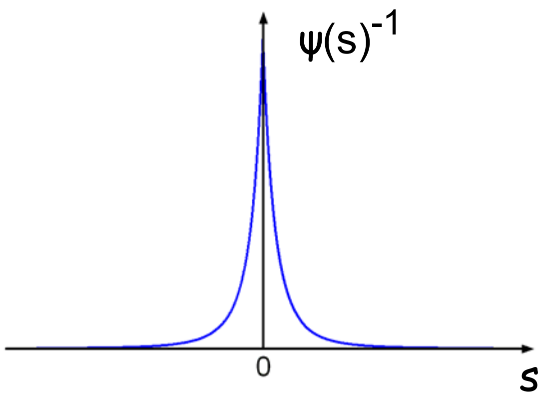
Three important class of problems with spatially-varying couplings:



(1) Systems with nearest neighbor coupling:

$$\psi(s) = \begin{cases} 1 & \text{if } |s| \leq d \\ 0 & \text{if } |s| > d \end{cases}$$

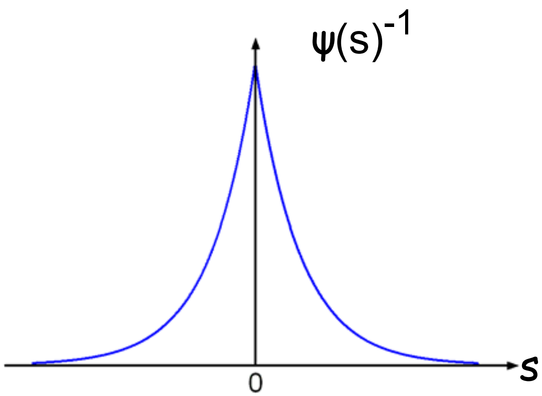
with $d > 0$



(2) Systems with exponentially decaying couplings:

$$\psi(s) = \zeta^{|s|}$$

with $\zeta > 1$



(3) Systems with algebraically decaying coupling:

$$\psi(s) = (1 + \alpha|s|)^{\beta}$$

with $\alpha, \beta > 0$

Properties of SD operators

Definition: Suppose Q is bounded and the coupling – characteristic function $\psi : R^{+} \to [1, \infty)$ is given. If $\tilde{Q}$ is bounded, then we say that Q is SD.

Theorem: sums, products and inverses of SD operators are SD.

Therefore, if A and B, Q , and R are SD

(1) Solution P of the Lyapunov Equation is SD:

$$A^{*}PA - P + Q = 0 \quad , \quad A^{*}P + PA + Q = 0$$

(2) Solution of the Algebraic Riccati Equation is SD:

$$A^{*}PA - P - A^{*}PB(R + B^{*}PB)^{-1}B^{*}PA + Q = 0 \quad \text{(DARE)}$$

$$A^{*}P + PA - PBR^{-1}B^{*}P + Q = 0 \quad \text{(CARE)}$$

(3) Solutions to finite horizon constrained quadratic optimization problems are SD.

Summary

Centralized solutions to finite and infinite horizon optimal control problems for spatially distributed systems has an inherent spatial locality.