CDS 202 Winter 2009 Solution Set 6

Problem 1 (MTA 5.1-1)

For $g \in GL(n, \mathbb{R})$ and $\xi, \eta \in L(\mathbb{R}^n, \mathbb{R}^n)$, by definition of the adjoint operator,

$$\mathrm{Ad}_g \xi = g\xi g^{-1}$$

and hence that

$$\begin{aligned} \mathrm{Ad}_g[\xi, \eta] &= g[\xi, \eta]g^{-1} = g(\xi\eta - \eta\xi)g^{-1} \\ &= g\xi g^{-1} g\eta g^{-1} - g\eta g^{-1} g\xi g^{-1} \\ &= [g\xi g^{-1}, g\eta g^{-1}] \\ &= [\mathrm{Ad}_g\xi, \mathrm{Ad}_g\eta]. \end{aligned}$$

Problem 2 (MTA 5.1-2)

It is clear that TG is a manifold. Hence, we only need to show that TG is a group and that the group multiplication is smooth. Assume that $(g, \tilde{g}) \in TG$ and $(h, \tilde{h}) \in TG$. The group multiplication is defined as $T\mu : TG \times TG \to TG$,

$$T\mu : ((g, \tilde{g}), (h, \tilde{h})) \mapsto (gh, T_h L_g \tilde{h} + T_g R_h \tilde{g}).$$

TG is a group, because the group multiplication as defined above satisfies:

- (Closure) Since $T_h L_g \tilde{h} + T_g R_h \tilde{g}$ is in $T_{gh}G$, we see that $(g, \tilde{g}) \cdot (h, \tilde{h}) \in TG$.
- (Associativity)

$$\begin{aligned} ((g, \tilde{g}) \cdot (h, \tilde{h})) \cdot (f, \tilde{f}) &= (gh, T_h L_g \tilde{h} + T_g R_h \tilde{g}) \cdot (f, \tilde{f}) \\ &= (ghf, T_f L_{gh} \tilde{f} + T_{gh} R_f (T_h L_g \tilde{h} + T_g R_h \tilde{g})) \\ &= (ghf, T_f L_{gh} \tilde{f} + T_h (R_f \circ L_g) \tilde{h} + T_g (R_f \circ R_h) \tilde{g}) \\ &= (ghf, T_f (L_g \circ L_h) \tilde{f} + T_h (L_g \circ R_f) \tilde{h} + T_g R_{hf} \tilde{g}) \\ &= (ghf, T_{hf} L_g (T_f L_h \tilde{f} + T_h R_f \tilde{h}) + T_g R_{hf} \tilde{g}) \\ &= (g, \tilde{g}) \cdot (hf, T_f L_h \tilde{f} + T_h R_f \tilde{h}) \\ &= (g, \tilde{g}) \cdot ((h, \tilde{h}) \cdot (f, \tilde{f})) \end{aligned}$$

- (Identity) The identity element is $(e, 0)$.

$$\begin{aligned} (g, \tilde{g}) \cdot (e, 0) &= (g, T_e L_g(0) + T_g R_e \tilde{g}) = (g, \tilde{g}) \\ (e, 0) \cdot (g, \tilde{g}) &= (g, T_g L_e \tilde{g} + T_e R_g(0)) = (g, \tilde{g}) \end{aligned}$$

- (Inverse) The inverse is $TI : (g, \tilde{g}) \mapsto (g^{-1}, -(T_e R_{g^{-1}} \circ T_g L_{g^{-1}})(\tilde{g}))$ (use Eq. (9.1.3) in the book).

$$\begin{aligned}
(g, \tilde{g}) \cdot (g, \tilde{g})^{-1} &= (g, \tilde{g}) \cdot (g^{-1}, -(T_e R_{g^{-1}} \circ T_g L_{g^{-1}})(\tilde{g})) \\
&= (e, T_{g^{-1}} L_g(-(T_e R_{g^{-1}} \circ T_g L_{g^{-1}})(\tilde{g})) + T_g R_{g^{-1}} \tilde{g}) \\
&= (e, -T_g(L_g \circ R_{g^{-1}} \circ L_{g^{-1}})\tilde{g} + T_g R_{g^{-1}} \tilde{g}) \\
&= (e, 0) \\
(g, \tilde{g})^{-1} \cdot (g, \tilde{g}) &= (g^{-1}, -(T_e R_{g^{-1}} \circ T_g L_{g^{-1}})(\tilde{g})) \cdot (g, \tilde{g}) \\
&= (e, T_g L_{g^{-1}} \tilde{g} + T_{g^{-1}} R_g(-(T_e R_{g^{-1}} \circ T_g L_{g^{-1}})(\tilde{g}))) \\
&= (e, T_g L_{g^{-1}} \tilde{g} - T_g(R_g \circ R_{g^{-1}} \circ L_{g^{-1}})\tilde{g}) \\
&= (e, 0).
\end{aligned}$$

Moreover the group operation is smooth since the group operation μ is smooth.

Problem 3

(a) (Nawaf Bou-Rabee) Let $\mathfrak{X}_R(G)$ denote the set of right invariant vector fields on G. We will show that the map $T_eG \to \mathfrak{X}_R(G)$ given by:

$$T_e R_g \cdot \xi = Y_\xi(g)$$

is right-invariant, smooth, and an isomorphism.

To show it is right-invariant consider:

$$T_g R_h \cdot Y_\xi(g) = T_g R_h \cdot T_e R_g \cdot \xi = T_e(R_g \circ R_h)\xi = T_e R_{hg} \cdot \xi = Y_\xi(hg)$$

Thus, $Y_\xi \in \mathfrak{X}_R(G)$.

Moreover, $\mathfrak{X}_R(G)$ is closed under the Lie bracket, i.e., for $X, Y \in \mathfrak{X}_R(G)$ then $[X, Y] \in \mathfrak{X}_R(G)$ since

$$(R_g)_*[X, Y] = [(R_g)_*X, (R_g)_*Y] = [X, Y]$$

which follows from the definition of right invariance ($(R_g)_*X = X$ and $(R_g)_*Y = Y$) and smoothness of R_g. Thus, $\mathfrak{X}_R(G)$ forms a Lie algebra.

To show it is isomorphic to T_eG consider the map $\zeta_1 : \mathfrak{X}_R(G) \to T_eG$ given by

$$Y_\xi \mapsto Y_\xi(e) = \xi \in T_eG$$

and the map $\zeta_2 : T_eG \to \mathfrak{X}_R(G)$

$$\xi \mapsto T_e R_g \cdot \xi \in \mathfrak{X}_R(G)$$

Since $\zeta_1 \circ \zeta_2 = \mathrm{id}_{T_eG}$ and $\zeta_2 \circ \zeta_1 = \mathrm{id}_{\mathfrak{X}_R(G)}$, the two spaces are bijective. Moreover these maps preserve linearity, hence the two spaces are isomorphic.

(b) (Nok Wingpiromsarn 2007) Since ϕ is the inversion map, we obtain

$$T_g\phi \cdot u = -T_g(R_{g^{-1}} \circ L_{g^{-1}}) \cdot u$$

for all $u \in T_gG$. Also, since X is left invariant, we have

$$(T_hL_g)X(h) = X(gh)$$

Consequently, we get

$$\begin{aligned}
(\phi_*X)(g) &= T_{g^{-1}}\phi \cdot X \cdot \phi^{-1}(g) \\
&= -T_{g^{-1}}(R_g \circ L_g) \cdot X(g^{-1}) \\
&\overset{\text{chain rule}}{=} -T_eR_g \cdot T_{g^{-1}}L_g \cdot T_eL_{g^{-1}} \cdot X(e) \\
&= -T_eR_g \cdot T_e(L_{g^{-1}} \circ L_g) \cdot X(e) \\
&= -T_eR_g \cdot X(e)
\end{aligned}$$

So $(\phi_*X)(e) = -T_eR_e(X(e)) = -X(e)$. Also,

$$\begin{aligned}
(\phi_*X)(gh) &= -T_eR_{gh} \cdot X(e) \\
&= -T_e(R_h \circ R_g) \cdot X(e) \\
&\overset{\text{chain rule}}{=} -T_gR_h \cdot T_eR_g \cdot X(e) \\
&= T_gR_h \cdot (-T_eR_g(X(e))) \\
&= T_gR_h \cdot (\phi_*X)(g)
\end{aligned}$$

Thus, by definition, $\phi_*(X)$ is the right invariant vector field. Since ϕ is a diffeomorphism, ϕ is an isomorphism between the set of left and right invariant vector fields on G.

Finally, we want to show that $X \mapsto \phi_*(X)$ gives a Lie algebra isomorphism between the. Since ϕ is a diffeomorphism, from Proposition 4.2.23, $\phi_*[X, Y] = [\phi_*X, \phi_*Y]$. Thus, ϕ preserves the Jacobi-Lie bracket of left-invariant vector fields, i.e., given left and right invariant vector fields: X_ξ, X_η and Y_ξ, Y_η, by the calculation above,

$$\phi_*[X_\xi, X_\eta] = \phi_*X_{[\xi,\eta]} = [\phi_*X_\xi, \phi_*X_\eta] = [Y_\xi, Y_\eta] \in \mathfrak{X}_R(G)$$

Problem 4 (MTA 5.2-1 (iii)-(v)) (Nok Wingpiromsarn 2007)

(iii) Using the identity,

$$\widehat{\omega_1}\widehat{\omega_2} - \widehat{\omega_2}\widehat{\omega_1} = \widehat{\omega_1\omega_2}$$

we get

$$\begin{aligned}[X_1, X_2](R) &= (\mathbf{D}X_2 \cdot X_1 - \mathbf{D}X_1 \cdot X_2)(R) \\ &= (\hat{\omega}_2\hat{\omega}_1 - \hat{\omega}_1\hat{\omega}_2)R \\ &= -\widehat{\omega_1 \times \omega_2}R\end{aligned}$$

(PS 2009: Alternatively, in the notation of P.298 of the book, $X_i(R) = \widehat{\omega}^i R = Y_{\widehat{\omega}^i}(R)$, so

$$\begin{aligned}[X_1, X_2](R) &= [Y_{\widehat{\omega}^1}, Y_{\widehat{\omega}^2}] \\ &= -Y_{[\widehat{\omega}^1, \widehat{\omega}^2]} && \text{(cf top of P.299)} \\ &= -[\widehat{\omega}^1, \widehat{\omega}^2]R \\ &= -(\widehat{\omega}^1\widehat{\omega}^2 - \widehat{\omega}^2\widehat{\omega}^1)R \\ &= -(\widehat{\omega^1 \times \omega^2})R && \text{by a straightforward calculation.)}\end{aligned}$$

(iv) $A \in \mathrm{SO}(3) \implies AA^T = A^T A =$ Identity. If A is also symmetric, then $AA = e =$ Identity. So the set of matrices in SO(3) that are also symmetric is given by

$$\{A \mid A^2 = \text{ Identity}, \det A = +1\}$$

From Corollary 5.2.8, A can be written as

$$A = B \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} B^T$$

where $B \in O(3)$. So we have

$$\begin{aligned} e &= AA \\ &= B \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}^2 B^T \\ &= B \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{bmatrix} B^T \\ B^T e B = e &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{bmatrix}\end{aligned}$$

Thus, we get that $\theta = n\pi, n \in \mathbb{Z}$. So the set of matrices in SO(3) that are also symmetric is given by

$$\text{Identity} \cup \left\{ B \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} B^T \;\middle|\; B \in O(3) \right\}$$

Note that these matrices correspond to a rotation through an angle $n\pi, n \in \mathbb{Z}$ about an axis $\mathbf{w}$ where $\mathbf{w}$ is an eigenvector of A with eigenvalue 1. These matrices have eigenvalues 1,1,1 or 1,-1,-1.

(v) Using the identity

$$(\hat{x}\hat{y})z = yx^T z - x^T yz = (yx^T - x^T ye)z$$

where e is the identity matrix, we get that

$$\begin{aligned}
\frac{1}{2}\operatorname{trace}(\hat{\omega}_1\hat{\omega}_2^T) &= -\frac{1}{2}\operatorname{trace}(\hat{\omega}_1\hat{\omega}_2) \\
&= -\frac{1}{2}\operatorname{trace}(\omega_2\omega_1^T - \omega_1^T\omega_2 e) \\
&= -\frac{1}{2}\left(\operatorname{trace}(\omega_2\omega_1^T) - \operatorname{trace}(\omega_1^T\omega_2 e)\right) \\
&= -\frac{1}{2}\left(\omega_1^T\omega_2 - 3\omega_1^T\omega_2\right) \\
&= \omega_1^T\omega_2 \\
&= \omega_1 \cdot \omega_2
\end{aligned}$$

Problem 5 (MTA 5.2-5)

(i) First, since $SO(n)$ and $\mathbb{R}^n$ are manifolds, it is clear that G is also a manifold. To show that G is a group, just notice that

- If (A, v) and (B, w) are in G (which means that $A, B \in SO(n)$), then AB is also in $SO(n)$ because $AB(AB)^T = ABB^T A^T = I$. In addition, $Aw + v$ is in $\mathbb{R}^n$. Thus $(A, v)(B, w)$ is in G.
- The associative property holds, i.e.,

$$\begin{aligned}
(A, v)((B, w)(C, y)) = (A, v)(BC, By + w) &= ABC, ABy + Aw + v) \\
&= ((A, v)(B, w))(C, y).
\end{aligned}$$

- The identity element is $(I, 0)$.
- The inverse of (A, v) is $(A^T, -A^T v)$, which is shown by $(A, v)(A^T, -A^T v) = (AA^T, -AA^T v + v) = (I, 0)$ and $(A^T, -A^T v)(A, v) = (A^T A, A^T v - A^T v) = (I, 0)$.

Thus, G is a group.

Finally, the group operation (which is just matrix - vector multiplications and addition) is smooth on $L(\mathbb{R}^n, \mathbb{R}^n) \times \mathbb{R}^n$, and hence on the submanifold G. So, we conclude that G is a Lie group.

(ii) Let $g = (A, w) \in G$. Define a submersion $f : G \to \mathbb{R}^n$ as follows:

$$f(g) = w.$$

Then $f(g) = 0$ defines a closed submanifold of G, which is exactly $SO(n) \times \{0\}$. Furthermore, $(A, 0)(B, 0) = (AB, 0)$, the identity $(I, 0)$, and the inverse $(A^T, 0)$ of $(A, 0)$ are all in $SO(n) \times \{0\}$, and the associative property follows immediately. Hence $SO(n) \times \{0\}$ is a subgroup of G.