CDS 140a Winter 2014 Homework 1

From MurrayWiki

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ACM 101/AM 125b/CDS 140a, Winter 2013 — Due: 15 Jan 2013

WARNING: This homework set has not yet been finalized. This banner will be removed when the problems are in their final form.

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. **Perko, Section 1.1, Exercise 3**: Find the general solution of the linear system

$$\begin{aligned} \dot{x}_1 &= x_1 \\ x_2 &= a x_2 \end{aligned}$$

where a is a constant. Sketch the phase portraits for $a = -1, a = 0$ and $a = 1$ and notice the qualitative structure of the phase portrait is the same for all $a < 0$ as well as for all $a > 0$, but that it changes at the parameter value $a = 0$, called a *bifurcation value*.

2. **Perko, Section 1.1, Exercise 6:**
(a) If $u(t)$ and $v(t)$ are sollutions of the linear system

$$\dot{x} = Ax,$$

prove that for any constants a and $b, w(t) = au(t) + bv(t)$ is a solution.
(b) For

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix},$$

find solutions $u(t)$ and $v(t)$ of $\dot{x} = Ax$ such that every solution is a linear combination of $u(t)$ and $v(t)$.

3. **Perko, Section 1.2, Exercise 6:** Let the $n \times n$ matrix A have real, distinct eigenvalues. Let $\phi(t, x)$ the the solution of the initial value problem

$$\dot{x} = Ax \qquad\qquad x(0) = x_0.$$

Show that for each fixed $t \in \mathbb{R}$,

$$\lim_{y_0 \to x_0} \phi(t, y_0) = \phi(t, x_0).$$

This shows that the solution $\phi(t, x_0)$ is a continuous function of the initial condition.

4. (Based on Perko, Section 1.3, Exercise 5, 6)
(a) For each matrix below, find the eigenvalues and eigenvectors of A and e^A:

$$\text{(i)} \begin{bmatrix} a & 0 \\ 0 & -b \end{bmatrix} \qquad \text{(ii)} \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \qquad \text{(iii)} \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$$

(remember to show the steps required for these [simple!] computations, don't just plug in values from MATLAB or Mathematica; see notes at the bottom of the page).

(b) Show that if x is the eigenvector of A corresponding to the eigenvalue λ, then x is also an eigenvector of e^A corresponding to the eigenvalue e^λ.

5. **Perko, Section 1.4, Exercise 4:** Given

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix},$$

compute the 3×3 matrix e^{At} and solve $\dot{x} = Ax$.

6. (Based on Perko, Section 1.4, Exercise 6) Let $E \subset \mathbb{R}^n$ an invariant subspace of $A : \mathbb{R}^n \to \mathbb{R}^n$ (i.e., for all $x \in E, Ax \in E$). Show that if $x(t)$ is the solution of the initial value problem

$$\dot{x} = Ax \qquad\qquad x(0) = x_0$$

with $x_0 \in E$, then $x(t) \in E$ for all $t \in \mathbb{R}$.

7. **Perko, Section 1.5, Exercise 1:** Use the theorem in Seciton 1.5 to determine if the linear system $\dot{x} = Ax$ has a saddle, node, focus or center at the origin and determine the stability of each node or focus:

$$\text{(a) } A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \text{(b) } A = \begin{bmatrix} \lambda & -2 \\ 1 & \lambda \end{bmatrix} \qquad \text{(c) } A = \begin{bmatrix} \lambda & 2 \\ 1 & \lambda \end{bmatrix}$$

(If your answer depends on the value of a parameter, make sure to describe all possible cases.)

8. **Perko, Section 1.6, Exercise 2:** Solve the initial value problem

$$\dot{x} = Ax \qquad\qquad x(0) = x_0$$

with

$$A = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Determine the stable and unstable subspaces and sketch the (3D) phase portrait. (Hint: see Figure 1 in Section 1.6 for an example of a 3D phase portrait.)

Notes:

- The problems are transcribed above in case you don't have access to Perko. However, in the case of discrepancy, you should use Perko (third edition) as the definitive source of the problem statement.
- There are a number of problems that can be solved using MATLAB. If you just give the answer with no explanation (or say "via MATLAB"), the TAs will take off points. Instead, you should show how the solutions can be worked out by hand, along the lines of what is done in the text book. It is fine to check everything with MATLAB.
- For numerical calculations, it is OK to use MATLAB to invert a matrix. But you should not use it to compute the matrix exponential and just put down the answer. Instead, show how to get the matrix exponential into a form in which the calculation can be done by hand (similar to what was done in lecture on 6 Jan) and then carry out the computation.
- For phase portraits, you should generate the diagram by hand and make sure to label any important features. Describe why the portrait looks as it does based on the relevant properties of the dynamical system (eg, eigenvalues of the A matrix).
- For the final exam, you will *not* be allowed to use MATLAB, Mathematica or similar programs, so make sure you understand what you are computing and drawing!
