CDS140a
Nonlinear Systems: Local Theory
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5 Gradient and Hamiltonian Systems

Here we discuss some special nonlinear ODE that have some very nice properties.

Definition: Let E be an open subset of $\mathbb{R}^{2n}$ and let $H \in C^2(E)$ where $H = H(x,y)$ with $x, y \in \mathbb{R}^n$. The system

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial y} \\ \dot{y} &= -\frac{\partial H}{\partial x}, \end{aligned}$$

where

$$\begin{aligned} \frac{\partial H}{\partial x} &= \left(\frac{\partial H}{\partial x_1}, \dots, \frac{\partial H}{\partial x_n}\right)^T \\ \frac{\partial H}{\partial y} &= \left(\frac{\partial H}{\partial y_1}, \dots, \frac{\partial H}{\partial y_n}\right)^T, \end{aligned}$$

is called a Hamiltonian system with n degrees of freedom on E.

5.1 Properties of Hamiltonian Systems

- The equilibrium points of the Hamiltonian system correspond to the critical points of H.
  - An equilibrium points (a_x, a_y) are called non-degenerate if the determinant second derivative of H evaluated at the equilibrium point is nonzero, i.e., $\left|\frac{\partial^2 H(a_x,a_y)}{\partial (x,y)^2}\right| \neq 0$.
- Hamiltonian systems are conservative, i.e., $H(x,y)$ remains constant along the trajetories.
  Proof:
  $$\frac{d}{dt}H = \frac{\partial H}{\partial x}\cdot\dot{x} + \frac{\partial H}{\partial y}\cdot\dot{y} = \frac{\partial H}{\partial x}\cdot\frac{\partial H}{\partial y} - \frac{\partial H}{\partial y}\cdot\frac{\partial H}{\partial x} = 0$$
  - H is called the first integral of the system.
  - The trajectories of the system lie on the level surfaces of $H(x,y) = \textit{constant}$.
- If the second derivative of H (a symmetric $2n \times 2n$ matrix) evaluated at an equilibrium point (a_x, a_y) has eigenvalues with positive real part (i.e., is positive definite), the the equilibrium point is (Lyapunov) stable.
  - Another way of stating this is that if $H(x,y) - H(a_x,a_y)$ is sign definite in a neighborhood of (a_x, a_y), then (a_x, a_y) is Lyapunov stable.
    Proof: $H(x,y) - H(a_x,a_y)$ is a local Lyapunov function.
- Many mechanical systems are described by Hamiltonians of the form
  $$H(x,y) = U(x) + \frac{1}{2}\sum_{i=1}^{n} y_i^2$$
  where the force field is dervied from the potential $U(x)$, i.e., $\ddot{x} = F = -\frac{\partial V}{\partial x}$, $y = \dot{x}$.
  - In this case, isolated local minima of U correspond to stable equilibrium points.
  - Isolated local maxima of U correspond to unstable equilibrium points.

Example 1: Spherical Pendulum

$$H(x,y) = \frac{1}{2}\left(x_1^2 + x_2^2 + y_1^2 + y_2^2\right)$$

which is equivalent to the uncoupled harmonic oscillators

$$\begin{aligned} \ddot{x}_1 + x_1 &= 0 \\ \ddot{x}_2 + x_2 &= 0 \end{aligned}$$

Example 2: (Hénon and Heiles)

$$\begin{aligned} H(x,y) &= U(x) + \frac{1}{2}\left(y_1^2 + y_2^2\right) \\ &= \frac{1}{2}\left(x_1^2 + x_2^2\right) + x_1^2 x_2 - \frac{1}{3}x_2^3 + \frac{1}{2}\left(y_1^2 + y_2^2\right) \end{aligned}$$

The potential function $U(x)$ has this critical points $(0,0)$(local minimum, hence stable), and $(0,1)$, $\left(\pm\frac{1}{2}\sqrt{3}, -\frac{1}{2}\right)$(no maximum or minimum achieved, can be shown that are unstable). The system is given by

$$\begin{aligned} \dot{x}_1 &= y_1 \\ \dot{x}_2 &= y_2 \\ \dot{y}_1 &= -x_1 - 2x_1x_2 \\ \dot{y}_2 &= -x_2 - x_1^2 + x_2^2 \end{aligned}$$

5.2 Properties of Hamiltonian Systems with 1 Degree of Freedom ($\mathbb{R}^2$)

We write $\frac{\partial H}{\partial x} = H_x$ and $\frac{\partial H}{\partial y} = H_y$.

- If an equilibrium point of the system is a focus then it is not a strict local maximum or minimum of H
- Nondegenerate equilibrium points of analytic planar Hamiltonian systems are either
  - a (topological) saddle iff it is a saddle of $H(x,y)$
  - a center iff it is a strict local maximum or minimum of $H(x,y)$
- If $H(x,y) = \frac{1}{2}y^2 + U(x)$, where $y = \dot{x}$, $\ddot{x} = -\frac{\partial V}{\partial x} = g(x)$, then the system is called a *Newtonian* system. $U(x)$ is the potential energy and $\frac{1}{2}y^2$ is the kinetic energy of the system.
  - Equilibrium points are given by $(x_0, 0)$ where x_0 is a zero of $g(x)$
  - If x_0 is a strict local maximum of analytic $U(x)$ then $(x_0, 0)$ is a saddle.
  - If x_0 is a strict local minimum of analytic $U(x)$ then $(x_0, 0)$ is a center.
  - If x_0 is a horizontal inflection point of $U(x)$ then $(x_0, 0)$ is a cusp.
  - Phase portrait is sysmmetric with respect to the x-axis.

Example 3: $H(x,y) = \frac{1}{2}y^2 + U(x)$, with $U(x) = \frac{1}{2}x^2 - \frac{a}{4}x^4$. Critical points given by $x - ax^3 = 0 \Rightarrow x = 0$, $x = \pm\sqrt{1/a}$.

- For $a \leq 0$ one fixed point (local minimum), center
- For $a > 0$, 3 fixed points; $x = 0$ local minimum (center), and $x = \pm\sqrt{1/a}$ local maxima (saddles).

Plot the phase portrait for $a = 1$.

5.3 Gradient Systems

Definition: Let E be an open subset of $\mathbb{R}^n$ and let $V \in C^2(E)$. The system

$$\dot{x} = -\text{grad}V(x)$$

where

$$\text{grad}V(x) = \left(\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n}\right)^T$$

is called a gradient system on E.

- The equilibrium points of the gradient system correspond to the critical points of V (grad$V(x) = 0$.
- At regular points of V (i.e., non-critical points) the gradient vector grad$V(x)$ is perpendicular to the level surfaces of $V(x) = \textit{constant}$.
- Strict local minima of V correspond to asymptotically stable equilibrium points of the system.
  Proof: Take $V(x) - V(a)$ as Lyapunov function (a is the equilibrium point of interest)

5.3.1 Planar Systems

- Nondegenerate critical points of a **planar analytic gradient system** (i.e. in $\mathbb{R}^2$) are either a saddle or a node.
  - if (x_0, y_0) is a saddle of V it is a saddle of the system.
  - if (x_0, y_0) is a strict local maximum of V it is an unstable node of the system.
  - if (x_0, y_0) is a strict local minimum of V it is a stable node of the system.

Definition: Given the planar system

$$\begin{aligned} \dot{x} &= P(x, y) \\ \dot{y} &= Q(x, y) \end{aligned}$$

the system ***orthogonal*** to this is defined by

$$\begin{aligned} \dot{x} &= Q(x, y) \\ \dot{y} &= -P(x, y) \end{aligned}$$

- The orthogonal of a Hamiltonian system is a gradient system.

Example 4 : Find the orthogonal the system of example 3 with $H(x, y) = \frac{1}{2}y^2 + U(x)$, with $U(x) = \frac{1}{2}x^2 - \frac{a}{4}x^4$. $V(x, y) = H(x, y)$, and critical points are:

- For $a \leq 0$ one critical point (strict local minimum) at $(0, 0)$ hence a asymptotically stable node.
- For $a > 0$, 3 critical points : $(0, 0)$ (strict local minimum) at $(0, 0)$ hence a asymptotically stable node., and $(\pm\sqrt{1/a}, 0)$ saddles.

Plot the phase portrait for $a = 1$.

Example 5 : Let $V(x, y) = (x + y)^2$.

$$\begin{aligned} \dot{x} &= -2(x + y) \\ \dot{y} &= -2(x + y) \end{aligned}$$

Fixed points given by $x = -y$ and are minima (nonisolated). Can show that they are stable (indeed, if we let $z = x + y$ then $\dot{z} = -4z$). Level sets of V are given by $x + y = \textit{constant}$. Plot the phase portrait. What is the orthogonal system?