# 9 An Example - Center Manifold and Limit Cycle

For those interested, here is an example that might illustrate some of the concepts we have studied. If you find this confusing, just ignore it completely.

**Exercise:** Determine the stability of the equilibrium point at the origin and the asymptotic behavior for

$$\begin{aligned} \dot{x}_1 &= -x_2 + x_1(1-y)y \\ \dot{x}_2 &= x_1 + x_2(1-y)y \\ \dot{x}_3 &= -x_3^3(1+y) \\ \dot{y} &= -y + x_1^2 + x_2^2 + 2(1-y)y^2 \end{aligned} \tag{14}$$

**Solution:** The equilibrium point at the origin in non-hyperbolic

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

so cannot use linearization to determine its stability. Not sure at this point if a Lyapunov function can be used. A next logical step would be to reduce the system (from a system on $\mathbb{R}^4$ to a system in $\mathbb{R}^3$) by looking at the flow on the center manifold.

Notice that the center manifold is defined by

$$y = h(x_1, x_2, x_3) = x_1^2 + x_2^2$$

*Check*: On the manifold

$$\begin{aligned} Dh(x)\cdot x &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \\ &= 2x_1\left(-x_2 + x_1\left(1-x_1^2-x_2^2\right)\left(x_1^2+x_2^2\right)\right) + 2x_1\left(x_1 + x_2\left(1-x_1^2-x_2^2\right)\left(x_1^2+x_2^2\right)\right) \\ &= 2\left(1-x_1^2-x_2^2\right)\left(x_1^2+x_2^2\right)^2 \\ \dot{y} &= -y + x_1^2 + x_2^2 + 2(1-y)y^2 \\ &= 2\left(1-x_1^2-x_2^2\right)\left(x_1^2+x_2^2\right)^2 \end{aligned}$$

The flow on the center manifold is given by

$$\begin{aligned} \dot{x}_1 &= -x_2 + x_1\left(1-x_1^2-x_2^2\right)\left(x_1^2+x_2^2\right) \\ \dot{x}_2 &= x_1 + x_2\left(1-x_1^2-x_2^2\right)\left(x_1^2+x_2^2\right) \\ \dot{x}_3 &= -x_3^3\left(1+x_1^2+x_2^2\right) \end{aligned} \tag{15}$$

Notice that the first two dimensions of reduced system (15) are decoupled from the third one, and indeed the flow of the third dimension is always converging (i.e., if $\left(\phi_t^1(x), \phi_t^2(x), \phi_t^3(x),\right)$ is a solution of (15), then $\lim_{t\to\infty}\phi_t^3(x) = 0$). To get a better intuition, let change to polar coordinates for the first two components ($x_1 = r\cos\theta,\ x_2 = r\sin\theta$)

$$\begin{aligned} \dot{r} &= r^3\left(1-r^2\right) \\ \dot{\theta} &= 1 \\ \dot{x}_3 &= -x_3^3\left(1+r^2\right) \end{aligned} \tag{16}$$

It appears to have a stable limit cycle at $r = 1$ (which can be shown using $\tilde{V}(r) = -\int_1^r a^3\left(1-a^2\right)da = -\frac{1}{4}r^4 + \frac{1}{6}r^6 + \frac{1}{12}$ ). So the origin is unstable and the system (15) has a stable limit cycle defined by $\Gamma = (\cos t, \sin t, 0)$. This (coupled with the center manifold theorem) implies that the system (14) should have a stable limit cycle defined by $\tilde{\Gamma} = (\cos t, \sin t, 0, 0)$. Let's see if we can prove those claims.

- Let's first prove that $\Gamma = (\cos t, \sin t, 0)$ is a stable limit cycle of (15) . Let use the Lyapunov function $V(x_1, x_2, x_3) = -\frac{1}{4}r^4 + \frac{1}{6}r^6 + \frac{1}{12} + x_3^2$, $r = \sqrt{x_1^2 + x_2^2}$. Clearly $V(\Gamma) = 0$, and $V > 0$ in a neighborhood of $\Gamma$. Also,

$$\begin{aligned} \frac{d}{dt}V &= -\left(r^3 - r^5\right)\dot{r} + 2x_3\dot{x}_3 \\ &= -r^6\left(1-r^2\right)^2 - 2x_3^4\left(1+r^2\right) \end{aligned}$$

and therefore $\dot{V} < 0$ in a neighborhood of $\Gamma$.

- Can we prove that $\tilde{\Gamma}= (\cos t, \sin t, 0, 0)$ is a stable limit cycle of (14) using $V_2(x_1, x_2, x_3, y) = -\frac{1}{4}r^4 + \frac{1}{6}r^6 + \frac{1}{12} + x_3^2 + y^2$, $r = \sqrt{x_1^2 + x_2^2}$? Clearly $V_2(\tilde{\Gamma}) = 0$, and $V_2 > 0$ in a neighborhood of $\tilde{\Gamma}$, but

$$\begin{aligned}\frac{d}{dt}V_2 &= -\left(r^2 - r^4\right)\left(x_1\dot{x}_1 + x_2\dot{x}_2\right) + 2x_3\dot{x}_3 + 2y\dot{y} \\ &= -\left(1 - r^2\right) r^4 \left(1 - y\right) y - 2x_3^4(1 + y) - 2y^2 + 2yr^2 + 4(1 - y)y^3 \\ &? \;\; 0\end{aligned}$$

Direct method does not appear to work. In this case, we use a two step process: (1) First show that the flow eventually converges to the center manifold globally; and (2) use $V$ on the center manifold to show convergence to the limit cycle.

1. *First show that we eventually converge on the center manifold globally.* Let $U(x, y) = \left(y - x_1^2 - x_2^2\right)^2$. Clearly $U = 0$ for $y = x_1^2 + x_2^2$ and $U > 0$ otherwise. Notice that

$$\begin{aligned}\dot{U} &= 2\left(y - x_1^2 - x_2^2\right)\left(\dot{y} - 2x_1\dot{x}_1 - 2x_2\dot{x}_2\right) \\ &= 2\left(y - x_1^2 - x_2^2\right)\left(-y + x_1^2 + x_2^2 + 2(1 - y)y^2 - 2\left(x_1^2 + x_2^2\right)(1 - y)y\right) \\ &= 2\left(y - x_1^2 - x_2^2\right)\left(-y + x_1^2 + x_2^2 + 2\left(y - x_1^2 - x_2^2\right)(1 - y)y\right) \\ &= -2\left(y - x_1^2 - x_2^2\right)^2\left(2y^2 - 2y + 1\right) \\ &= -2\left(y - x_1^2 - x_2^2\right)^2\left(y^2 + (y - 1)^2\right)\end{aligned}$$

   and therefore $\dot{U} < 0$ for $y \neq x_1^2 + x_2^2$ and 0 otherwise. So trajectories eventually converge to the center manifold. Notice we just proved global convergence (rather than use the center manifold theorem to get local convergence).

2. *Use $V$ on the center manifold to show convergence to the limit cycle.* Need to be a little careful since the flow does not actually reach the center manifold (unless it starts on it), but the idea is there (can show that

$$\begin{aligned}\frac{d}{dt}V(x_1, x_2, x_3) &= -\left(r^2 - r^4\right)\left(x_1\dot{x}_1 + x_2\dot{x}_2\right) + 2x_3\dot{x}_3 + 2y\dot{y} \\ &= -\left(1 - r^2\right) r^4 \left(1 - y\right) y - 2x_3^4(1 + y)\end{aligned}$$

   is negative definite if we start near the center manifold, i.e., for $y = x_1^2 + x_2^2 + \epsilon = r^2 + \epsilon$, $\epsilon$ small.)

So successive applications of Lyapunov functions $U$ and $V$ give us that all trajectories (except for the one at the origin) converge to $\tilde{\Gamma}$.