Lecture Summary: Estimation and Control over Networks

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Motivation: To fully solve the networked control problem, problems that involve multiple sensors, communication channels, controllers and processes need to be considered. We briefly review the known results for the analog erasure channel and the digital noiseless channel models when (i) multiple sensors observe the same process and transmit information to the controller, and (ii) the sensor transmits information to the controller over a network of communication channels with an arbitrary topology. The multiple controller case involves issues of dual purpose control and is in general difficult to solve.

Analog Erasure Channel Model: Both the one-block and two-block design problems have been discussed for the case of a single sensor transmitting over an analog erasure channel to the controller. Those results can be generalized in the following two directions.

A Network of Communication Channels: Consider the LQG formulation when the sensor communicates with the controller across a network consisting of multiple communication channels connected according to an arbitrary topology. The sensor and the controller then form two nodes of a network each edge of which represents a communication channel. The erasure events on the channels are assumed to be independent of each other, for simplicity. The one-block design problem is identical to that for the one channel case, with the erasure probability as some function of the reliability of the network. This can lead to poor performance, since the reliability may decrease quickly as the network size increases.

The two-block design paradigm permits the nodes of the network to process the data prior to transmission, and hence achieve much better performance. The design problem can be solved in a manner similar to the single channel case. A separation principle holds if the actuator can transmit an acknowledgment to the controller. The optimal performance is then achieved by solving an optimal estimation problem and utilizing the LQR control law. The upper bound on estimation performance is provided by the strategy of every node transmitting every measurement it has access to at each time step. However, the same estimate is calculated at the decoder if the sensor transmits an estimate of the state at every time step and every other node (including the decoder) transmits the latest estimate it has access to from either its neighbors or its memory. This algorithm is optimal, although recursive. For the control problem, the effect of control inputs and measurements can be separated in an identical fashion as the single channel case. This ensures that the intermediate nodes do not require access to the control inputs. Moreover, as long as the closed loop system is stable, the quantities transmitted by various nodes are also bounded.

The stability and performance analysis are more complicated. As an example, a necessary and sufficient stability condition is that the inequality $p_{\text{maxcut}}\rho(A)^2 < 1$ holds, where $\rho(A)$ is the spectral radius of A and p_{maxcut} is the max-cut probability evaluated as follows. Generate cut-sets from the network by dividing the nodes into two sets - a source set containing the sensor, and a sink set containing the controller. For each cut-set, obtain the cut-set probability by multiplying the erasure probabilities of the channels from the source set to the sink set. The max-cut probability is the maximum such cut-set probability. The necessity of the condition follows by recognizing that the channels from the source set to the sink set need to transmit data at a high enough rate even if the channels within each set are assumed not to erase any data. The sufficiency of the condition follows by using the Ford-Fulkerson algorithm to reduce the network into a collection of parallel paths from the sensor to the controller such that each path has

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links with equal erasure probability and the product of these probabilities for all paths is the max-cut probability.

Multiple Sensors: If the process is observed using multiple sensors that transmit data to a controller, then the problem again reduces to an estimation problem. The two-block design for the estimation problem is similar to track-to-track fusion problem, which is open for general cases. The optimal encoding scheme, in general, is not to transmit local estimates because of the correlation introduced by the process noise. If erasure probabilities are zero, or process noise is not present, then the optimal encoding schemes are known and will be covered in the next lecture. Another case for which the optimal encoding schemes are known is when the estimator sends back acknowledgments to the encoders.

Transmitting local estimates does not, however, reduce the stability region as compared to the strategy of sensors transmitting all the measurements at each time step. The latter strategy is optimal, but requires an increasing amount of data transmission. As an example, the necessary and sufficient stability conditions for the two sensor case are given by

$$p_1 \rho(A_1)^2 < 1$$
$$p_2 \rho(A_2)^2 < 1$$
$$p_1 p_2 \rho(A_3)^2 < 1,$$

where p_1 and p_2 are erasure probabilities from sensors 1 and 2 respectively, $\rho(A_1)$ is the spectral radius of the unobservable part of the matrix A from the second sensor, $\rho(A_2)$ is the spectral radius of the unobservable part of the matrix A from the first sensor, and $\rho(A_3)$ is the spectral radius of the observable part of the matrix A from both the sensors.

Digital Noiseless Channels: Similar arguments hold for the digital noiseless channel model. Performance optimal encoders are not available even for the single sensor or channel case. Thus, two-block design for optimal stability conditions is considered.

A Network of Communication Channels: A max-flow min-cut like theorem again holds. The stability condition now becomes that for any cut-set

$$\sum R_j > \sum_{\text{all unstable eigenvalues}} \log_2(\lambda_i),$$

where $\sum R_j$ is the sum of rates over channels joining the source set to sink set for any cut-set.

Multiple Sensors: For every sensor i, define a rate vector $\{R_{i_1}, R_{i_2}, \cdots, R_{i_N}\}$ corresponding to the N modes of the system. If a mode j cannot be observed from the sensor i, set $R_{i_j} = 0$. For stability, the condition

$$\sum_i R_{ij} \geq \max(0, \lambda_j,$$

for every mode j must be satisfied. All such rate vectors stabilize the system.