Lecture Summary: Networked Control with Digital Noiseless Channels

Vijay Gupta* and Richard M. Murray †

Channel Description: A digital noiseless channel can support one symbol $s(k)$ from a finite alphabet $\mathcal{S}$ of cardinality $M \geq 1$, at each time k. The symbol is received without error at the decoder. The channel is said to support a (bit) rate $R = \log_2 M$ bits per sample at every time step.

One Block Design: The one block design problem for digital noiseless channel is largely open. If the rate is not too low, an approximation to assume that the quantization error is an additive white noise that is uncorrelated with the present process state can be made to solve the problem. The variance of the noise is determined by the quantizer structure. As an example, if a uniform quantizer with step size δ is used, the noise variance is assumed to be equal to $\frac{\delta^2}{12}$. In reality, quantization noise is not white, or uncorrelated with the process state. Thus, this approximation fails to capture analytic results such as the existence of chaotic trajectories in the presence of quantization, or a minimum bit rate required for stability.

Two Block Design: Consider the process

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + w(k) \\ y(k) &= Cx(k) + v(k), \end{aligned}$$

where $w(k)$ and $v(k)$ are bounded deterministic unknown noise sequences. The encoder, situated at the sensor, transmits the symbol $s(k) = \gamma(k, y(0), \cdots, y(k), s(0), \cdots, s(k-1)) \in \mathcal{S}$. There are no memory or processing constraints assumed. The decoder calculates the control input $u(k) = \delta(k, s(0), \cdots, s(k-d))$, where d is a constant that models the delay introduced by the channel. If an encoder-decoder pair exists such that the cost $J = \limsup_{k\to\infty}\{\|x(k)\|\}$ as the initial state and the process noises are varied remains bounded, the system is said to be stabilizable. Define the intrinsic entropy H of the process as $H = \log_2|\det(A_u)| = \sum_i \log_2(\max(\lambda_i(A), 0))$, where A_u is the unstable part of A and $\lambda_i(A)$ is the i-th eigenvalue of A.

Data Rate Theorem: The data rate theorem states that for causal encoder and decoder pairs with rate R,

- If $R \leq H$ and the process noise has non-zero support, then the process is not stabilizable.
- If $R > H$, then for any encoder and decoder,

$$\limsup_{k\to\infty} \|x(k)\| \geq \frac{\beta^{-1/n}\lambda(\mathcal{W})^{1/n}}{1 - 2^{-(R-H)/n}},$$

where β is the volume of an n-dimensional sphere of unit radius (where $x(k) \in \mathbf{R}^n$), and $\lambda(\mathcal{W})$ is the support of the set over which the process noise can vary.

The basic idea of the proof is to calculate the evolution of the volume of uncertainty in $\|x(k)\|$ by considering the gain due to the unstable eigenvalues, and reduction due to $s(k)$. The rate R does not need to be constant at every time step, but merely a long term average rate.

Tightness of Bounds: There are two questions about the tightness of the bounds above. The bound $R > H$ on data rates required for stability is tight. It is indeed possible to stabilize the system for $R = H + \varepsilon$ for any $\varepsilon > 0$. However, the bound on the state for $R > H$, turns out to be tight only for scalar plants.

*Department of Electrical Engineering, University of Notre Dame, vgupta2@nd.edu
†Control and Dynamical Systems, California Institute of Technology murray@caltech.edu

Stochastic Plants: Similar questions can be asked with respect to moment stability in stochastic systems, i.e., when the initial condition and noises are random variables with possibly infinite support. Similar arguments as the previous case, but considering the entropy power, yield identical conditions on rate for stability. However, if there are memory restrictions on the encoder, then moment stability cannot be achieved for noise with infinite support (while it can be for compact support). For noise with infinite support, a quantizer that adapts its range is required. Moreover, the adaptation parameter can have infinite number of values.

Asymptotic Stability: If there is no noise in the system, asymptotic stability may be possible. The basic result is that for an encoder and decoder with finite memory, asymptotic stability can not be achieved. It may be necessary to aim for practical stability in which the state enters a region of the state space and stays inside that region for all future time. Alternatively, encoder and decoder with infinite memory can be used, e.g., quantizers that zoom in as state moves into a neighborhood close to origin.

Performance: Encoders and decoders that achieve optimal performance are largely unknown. Since the applied control input is known to the controller, certainty equivalence holds. However, because of the non-linearity introduced by the quantizer, the function of state used in the control law is no longer the conditional mean based on measurements and control inputs. An alternate distortion metric needs to be minimized. However, the metric depends also on the input matrix, and hence the control and estimation parts are not fully decoupled.