An Information Theoretic Viewpoint to Performance Bounds of Feedback Systems: Optimality Results and Open Problems

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Outline

- Motivation for the study of fundamental limits in Engineering.

- Networked control systems as a test-bed for research in control and communications.

- Brief description of Bode's integral formula in discrete time.

-Derivation of a conservation law and its relation to common performance measures. (Relation to Bode's integral)

- A universal bound of disturbance attenuation in the presence of finite capacity feedback.

- Extension of Bode's integral in the presence of finite capacity disturbance preview.

- Conclusions

Importance of Fundamental Limits of Performance

Eliminate infeasible specifications

Proving optimality (Inequalities)

Design

Create Benchmarks

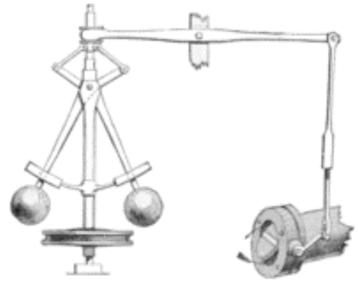
Applied Research

Unveil Underlying Fundamental Phenomena

Science

Two Examples in Engineering

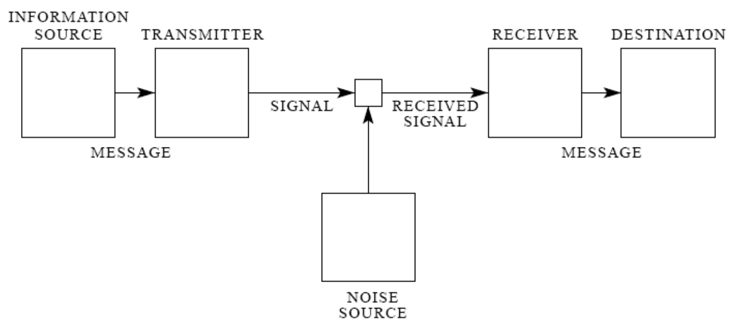
H. W. Bode



Linear Time-Invariant Feedback Systems

Sensitivity to external excitation

C. Shannon



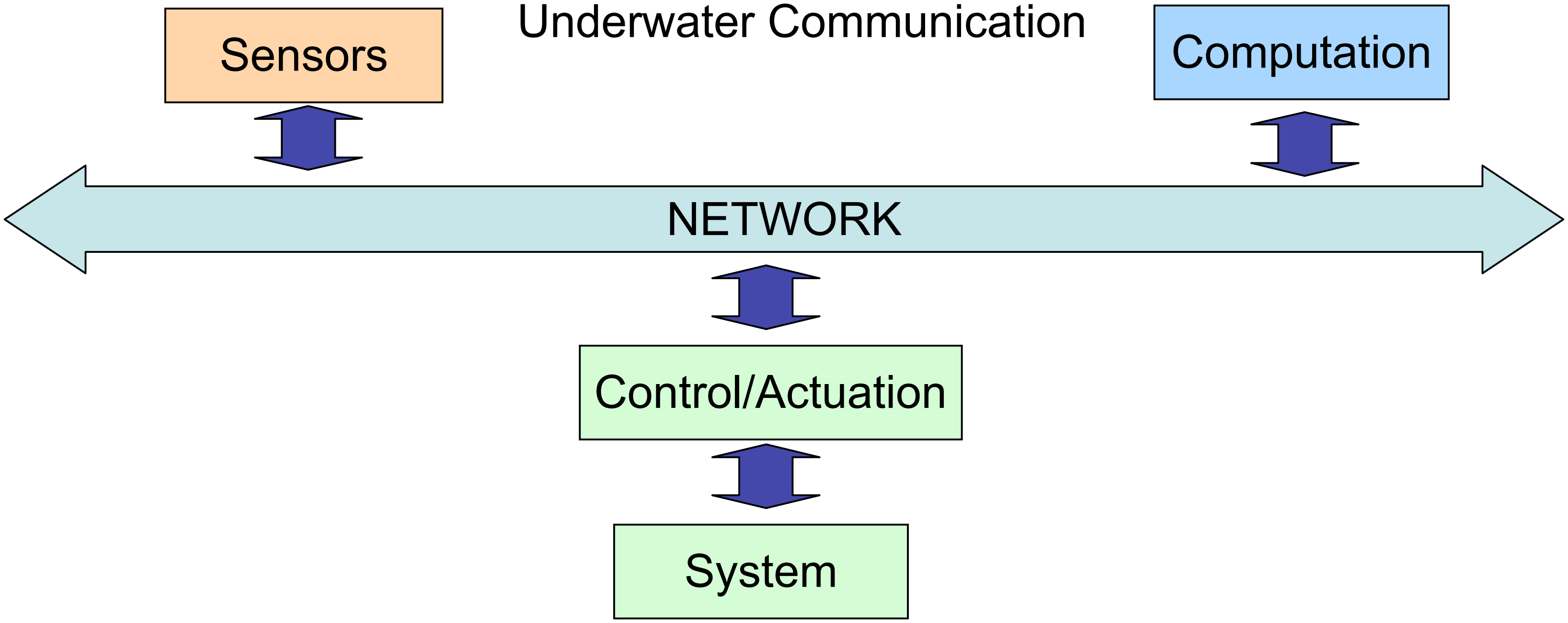
(Extracted from: A Mathematical Theory of Communication)

Maximum rate of reliable information transfer

Motivation for studying communication and controls

Wireless networks

Underwater Communication

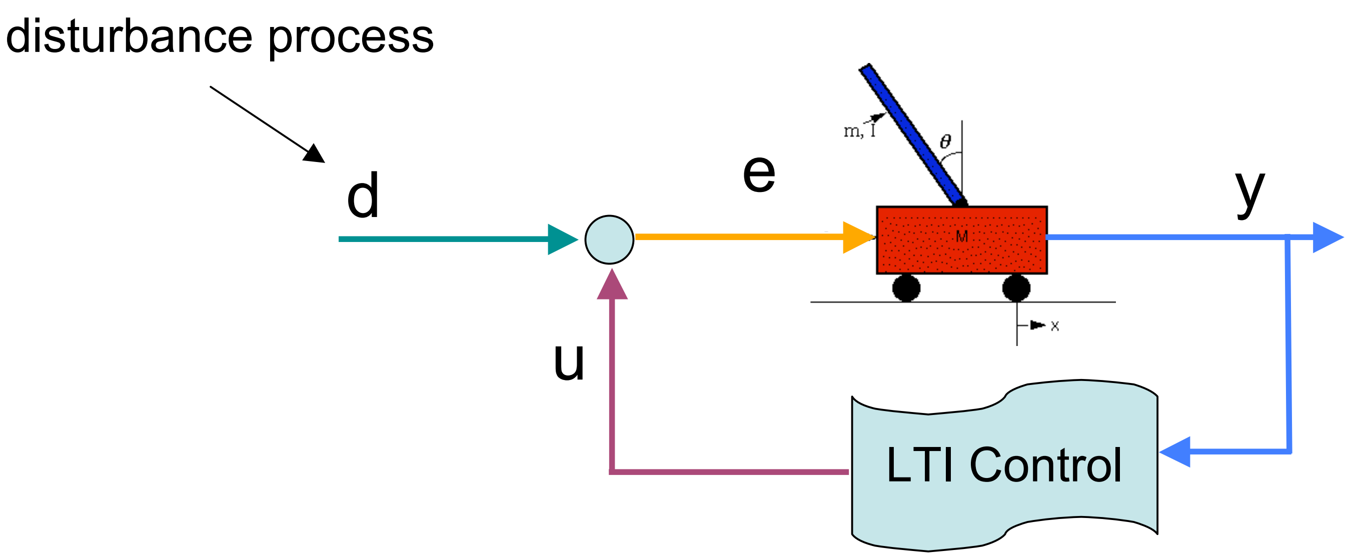


Distributed Remote Control

Multi-agent Applications

Bode’s Integral Limitation

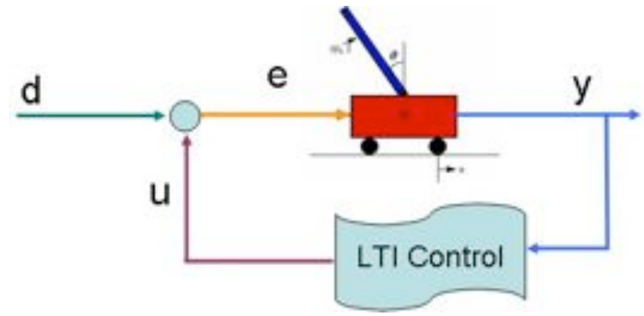
Linear Feedback Scheme:

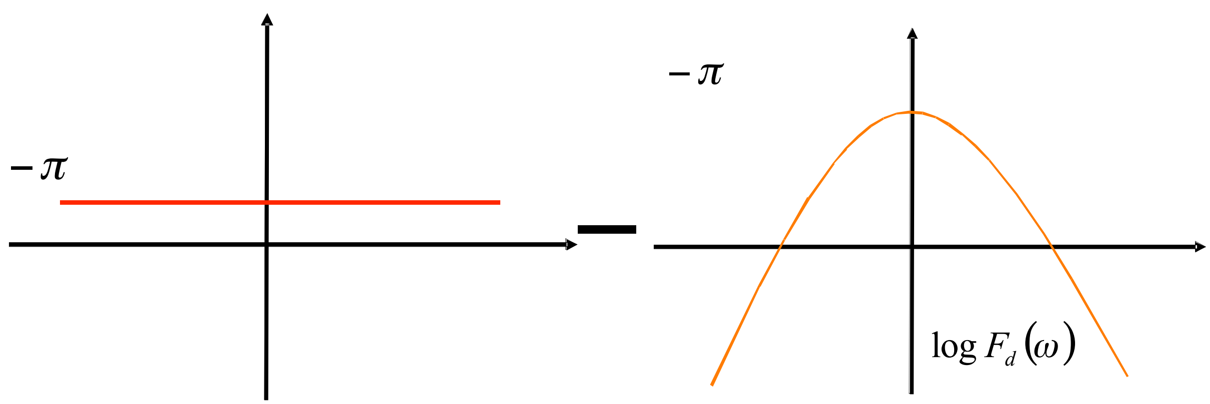


What can we do reject disturbances?

Bode's Integral Limitation: A sensitivity-like function

Measuring performance …



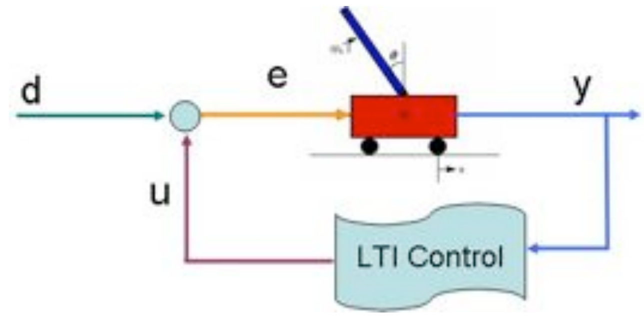


Spectrum of the error

Spectrum of the disturbance

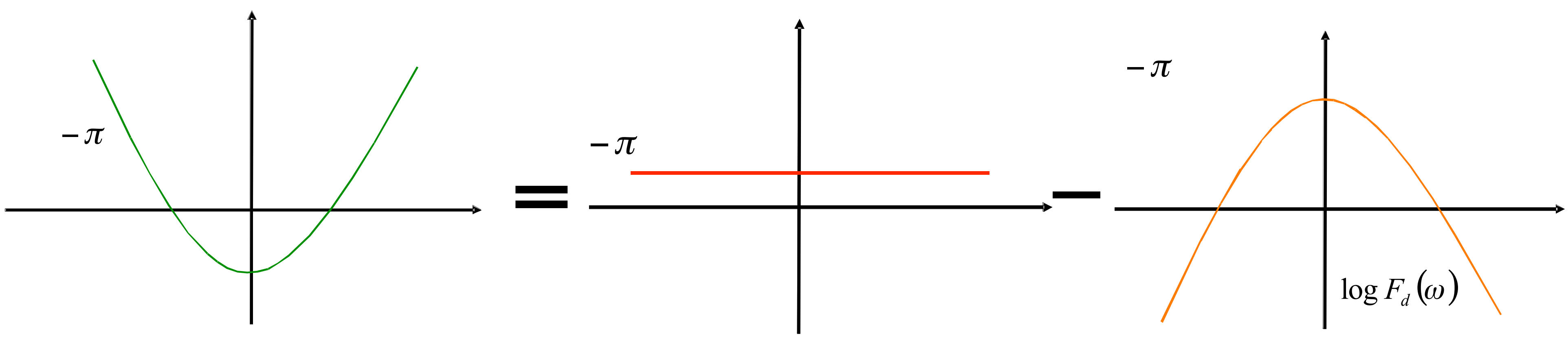
Bode’s Integral Limitation: A sensitivity-like function

Measuring performance …



$$S^2_{e,d}(\omega) = \frac{F_e(\omega)}{F_d(\omega)}$$

Sensitivity-like Function

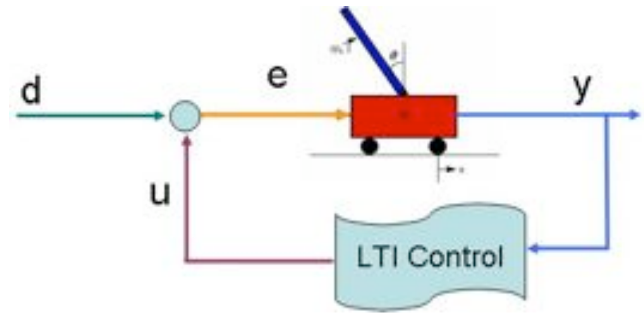


Log-sensitivity (square) — Spectrum of the error — Spectrum of the disturbance

Bode's Integral Limitation

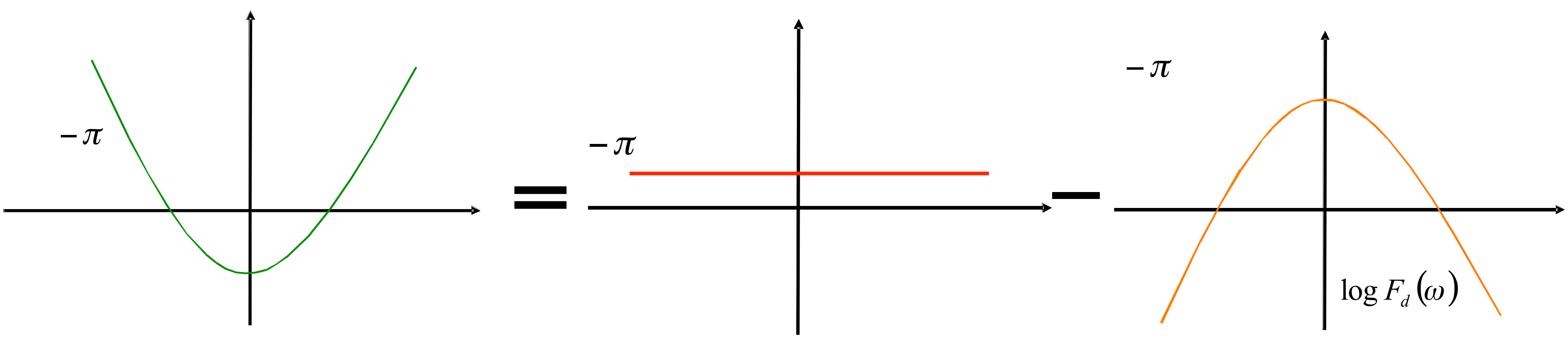
We can't push the log-average down

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log S_{e,d}^{2}(\omega)\,d\omega \ge 2\sum_{unstable\ poles}\log\left|pole_i\right|$$



Bode (1945) for continuous time

S. Hara (1989) for discrete-time

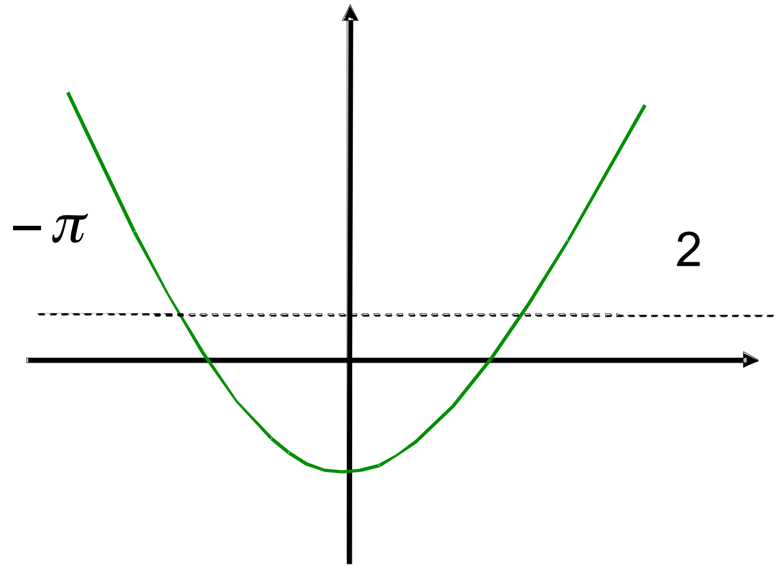


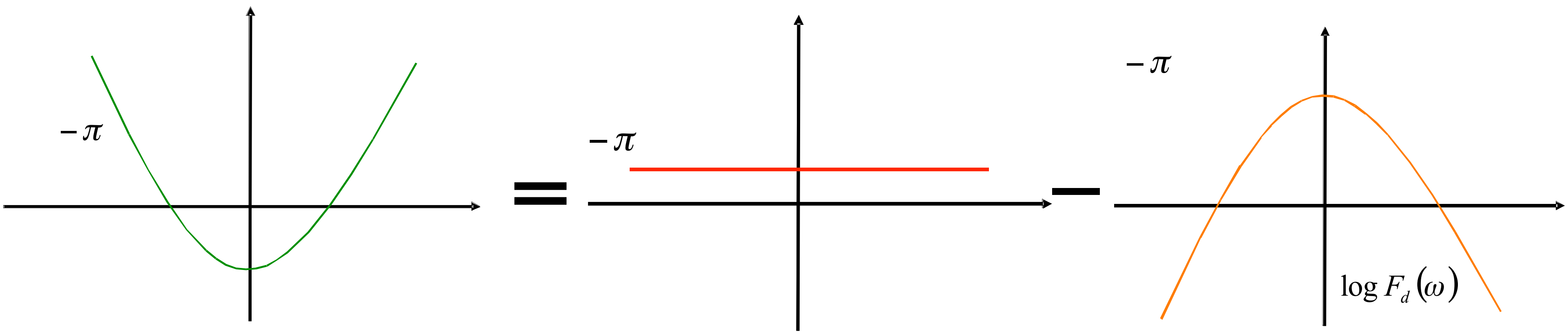
Log-sensitivity (square) | Spectrum of the error | Spectrum of the disturbance

Bode's Integral Limitation

Water-bed effect
(Conservation law)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log S_{e,d}^{2}(\omega)\,d\omega \geq 2 \sum_{unstable\ poles} \log\left|pole_i\right|$$





Log-sensitivity (square)

Spectrum of the error

Spectrum of the disturbance

Bode's Integral Limitation: Extensions

Extensions: multivariable, time-varying …

- Freudenberg (88), Seron, Braslavsky, Goodwin (97)

Information Theoretic Interpretation: Extensions to classes of Non-Linear Systems

- Zang and Iglesias, (96) and Jonckheere (92)

Deterministic approach

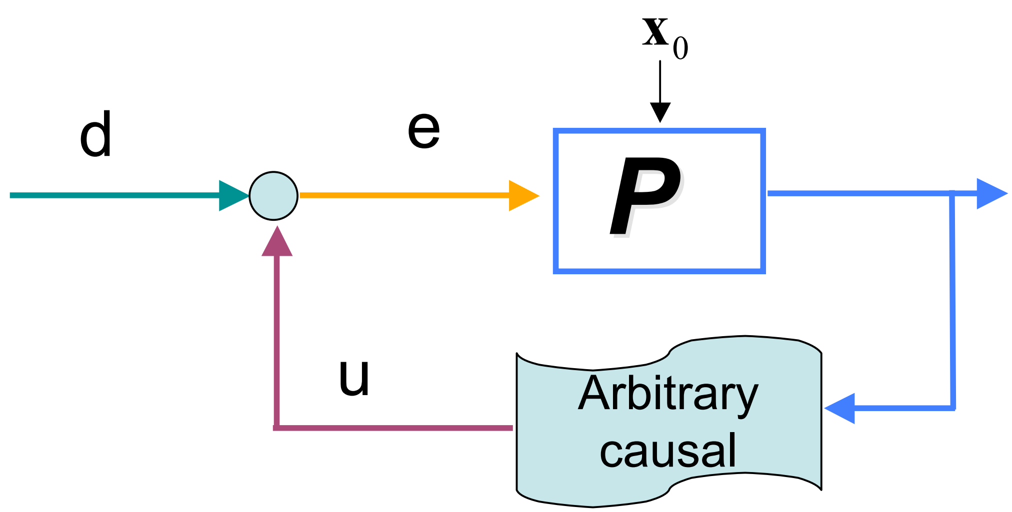
- Yi, Goncalves, Ingals, Sauro, Doyle (in preparation)

Using theories related to Bode's integral to design coding schemes.

- Elia, N.

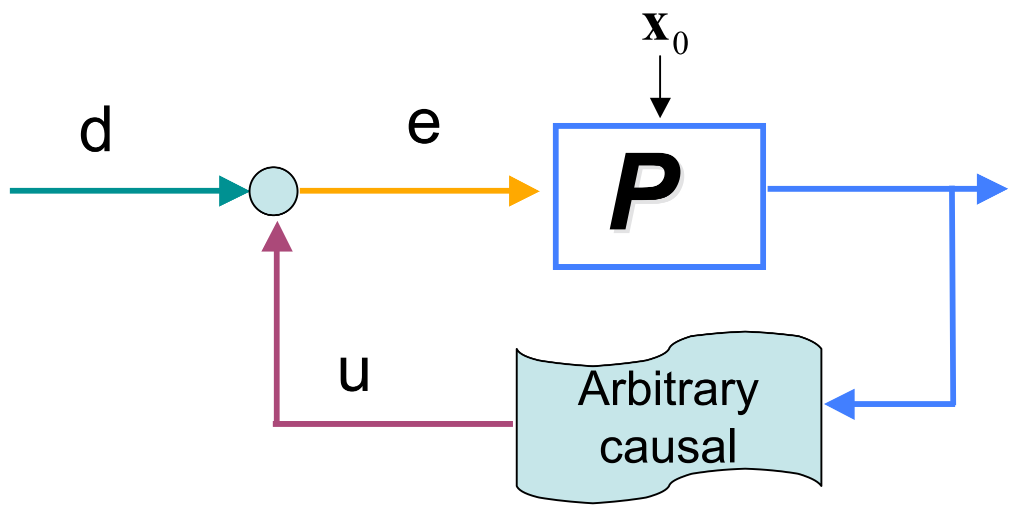
New information theoretic interpretation and extension for arbitrary feedback

- Martins & Dahleh (2004), Martins Ph.D. dissertation
- Martins, Dahleh and Doyle (2005)



- ***P*** is finite dimensional, linear, time-invariant, single input and single output

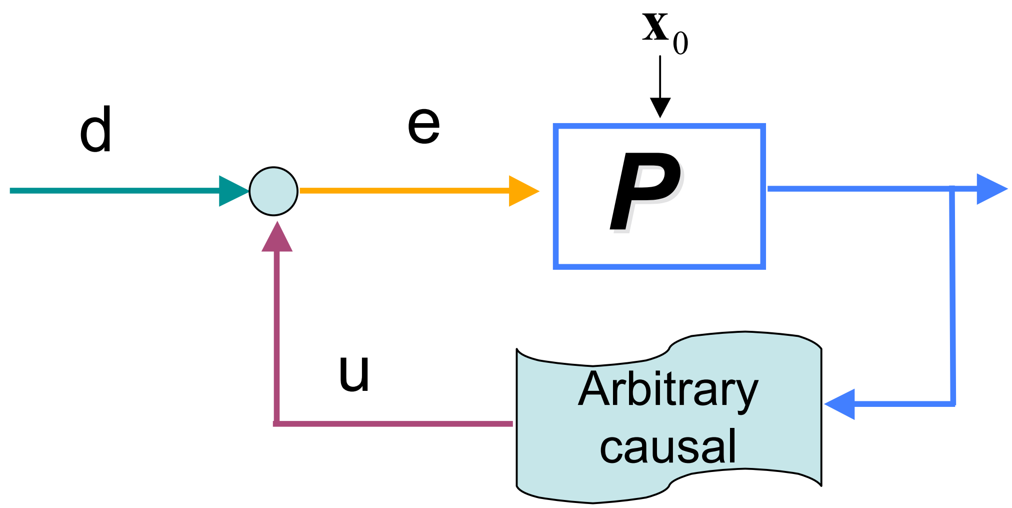
- if the state of ***P*** is represented by $\mathbf{x}(k) \in \Re^n$ then the initial state $\mathbf{x}_0 = \mathbf{x}(0)$ is a random variable.



Question

Assuming that **d** and **e** are asymptotically stationary, will the following hold for arbitrary causal feedback ?

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log S_{e,d}^2(\omega)\,d\omega \geq 2 \sum_{unstable\ poles} \log|pole_i| \qquad S_{e,d}^2(\omega) = \frac{F_e(\omega)}{F_d(\omega)}$$



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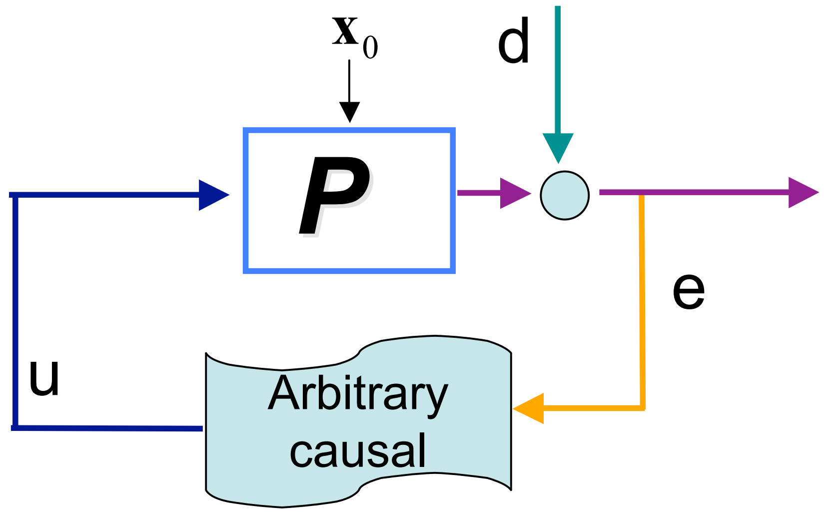
Answer:

Using optimal linear one-step prediction theory (Kolmogorov), we can show that the answer is **yes**. This result holds regardless of the distribution of **d**.



What happens if the disturbance enters at the output (tracking) ?

Assuming that **d** and **e** are asymptotically stationary, will the following hold for arbitrary causal feedback ?

The answer is: it depends on the distribution of **d**. In general the answer is **NO**.

Bode’s Integral Limitation: What are we searching for ?

We seek a Theory that:

• Explains the fundamental limits of feedback for different configurations in a unified fashion.

(Information flow interpretation)

• Quantifies the role of the probability distribution of the disturbance.

(maximum entropy principle)

• Allows for the analysis of other frameworks relevant for networked control. Such as the introduction of side information and information-rate constraints.

(Algebraic properties of mutual information)

Using information theory …

Fundamental limits of feedback: An Information Theoretic Approach

Differential Entropy

Consider the stochastic process:

$$\mathbf{a}^k = \left(\mathbf{a}(0), \cdots, \mathbf{a}(k)\right)$$

$$h\left(\mathbf{a}^k\right) = -\int_{\Re^k} p_{a^k}(\gamma) \log_2 p_{a^k}(\gamma) d\gamma$$

(Differential Entropy)

Differential Entropy

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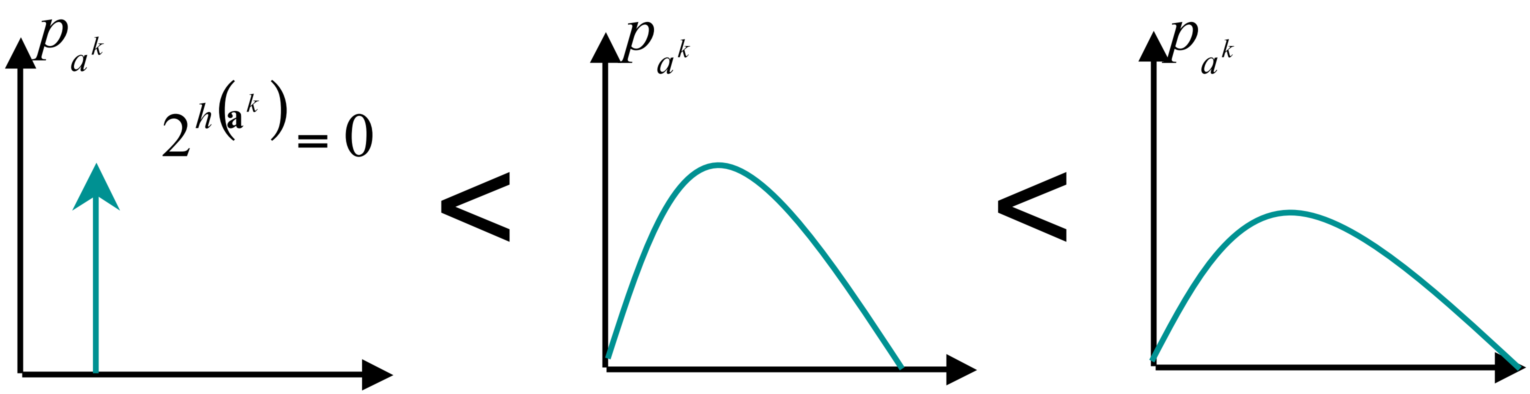

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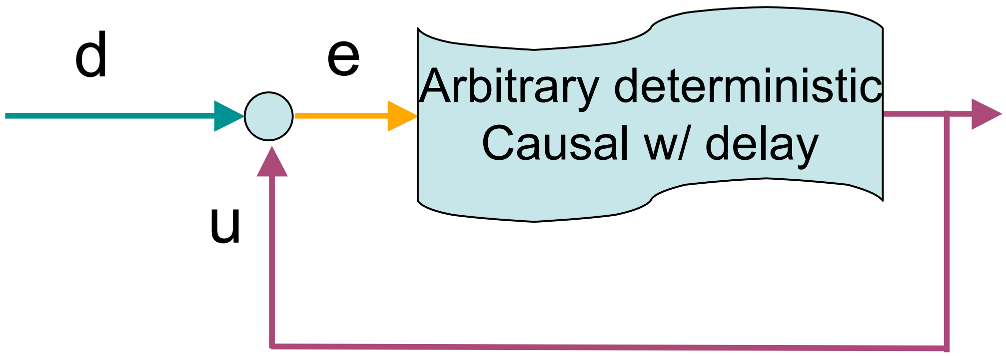
$$\mathbf{a}^k = \left(\mathbf{a}(0), \cdots, \mathbf{a}(k)\right)$$

$$h\left(\mathbf{a}^k\right) = h\left(\mathbf{a}(k) \mid \mathbf{a}^{k-1}\right) + h\left(\mathbf{a}^{k-1}\right)$$

$$h_\infty(\mathbf{a}) = \limsup_{k\to\infty} \frac{h\left(\mathbf{a}^k\right)}{k} = \limsup_{k\to\infty} \frac{\sum_{i=1}^{k} h\left(\mathbf{a}(i) \mid \mathbf{a}^{i-1}\right) + h\left(\mathbf{a}(0)\right)}{k}$$

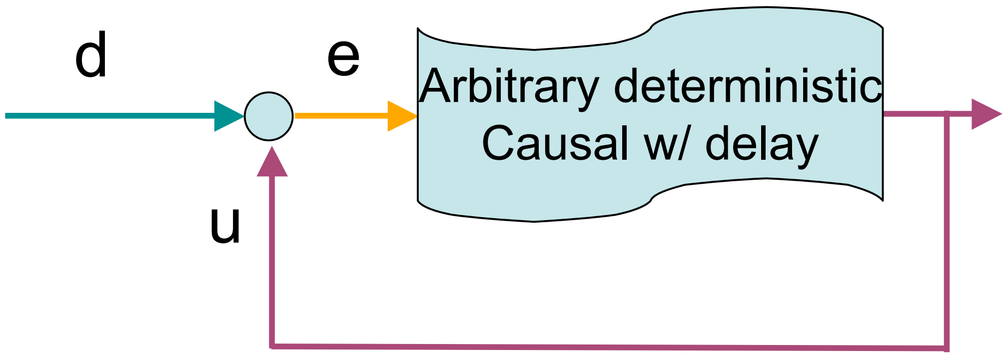
(Entropy rate)

A conservation law using differential entropy



$$h(\mathbf{d}^k) = h(\mathbf{e}^k)$$

A conservation law using differential entropy



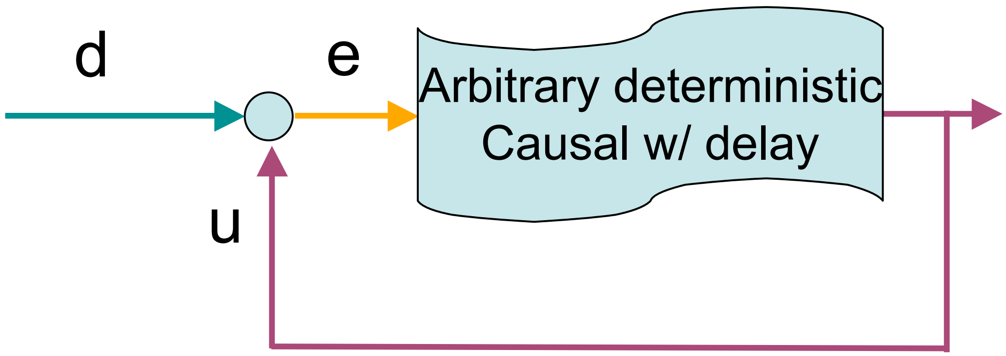
$$h(\mathbf{d}^k) = h(\mathbf{e}^k)$$

Proof:

$$h(\mathbf{d}(k) \mid \mathbf{d}^{k-1}) = h(\mathbf{d}(k) \mid \mathbf{u}^k, \mathbf{d}^{k-1})$$

$$h(\mathbf{d}(k) \mid \mathbf{u}^k, \mathbf{d}^{k-1}) = h(\mathbf{e}(k) \mid \mathbf{u}^k, \mathbf{e}^{k-1})$$

A conservation law using differential entropy



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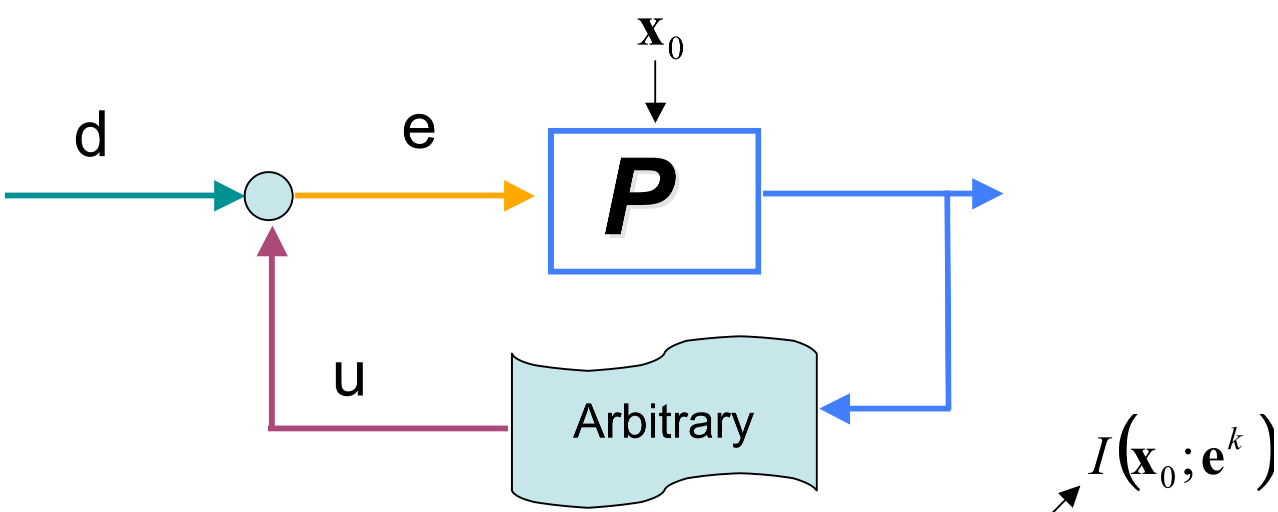
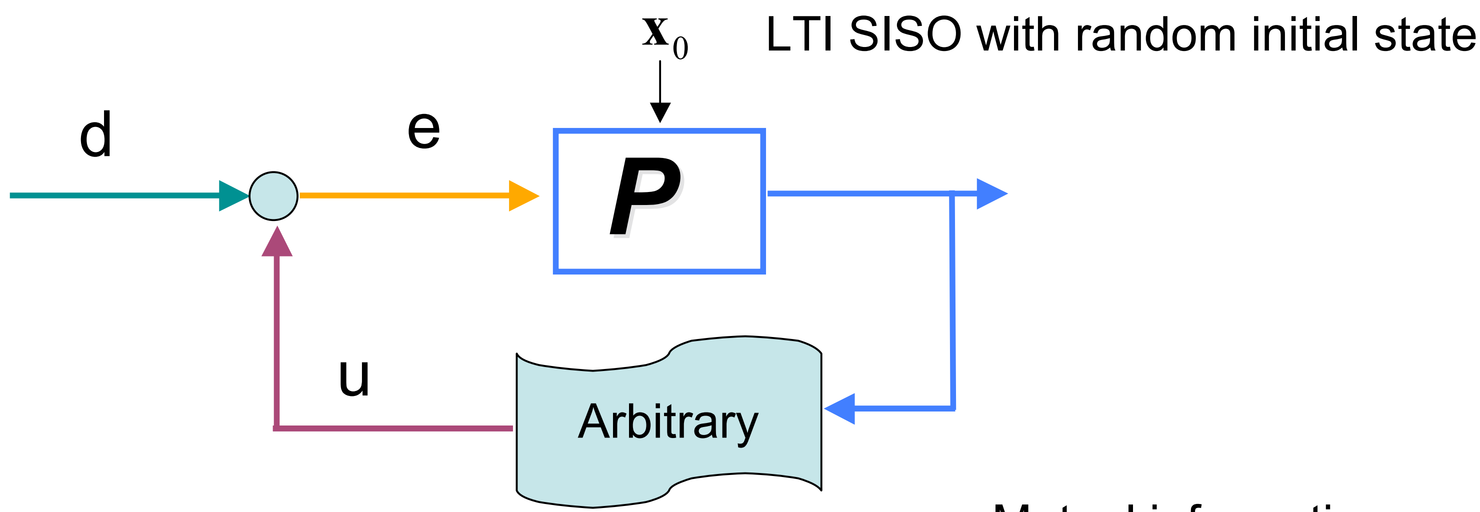


$$h(\mathbf{e}^k) \geq h(\mathbf{d}^k) + h(\mathbf{x}_0) - h(\mathbf{x}_0 \mid \mathbf{e}^k)$$

A conservation law using differential entropy



Mutual information

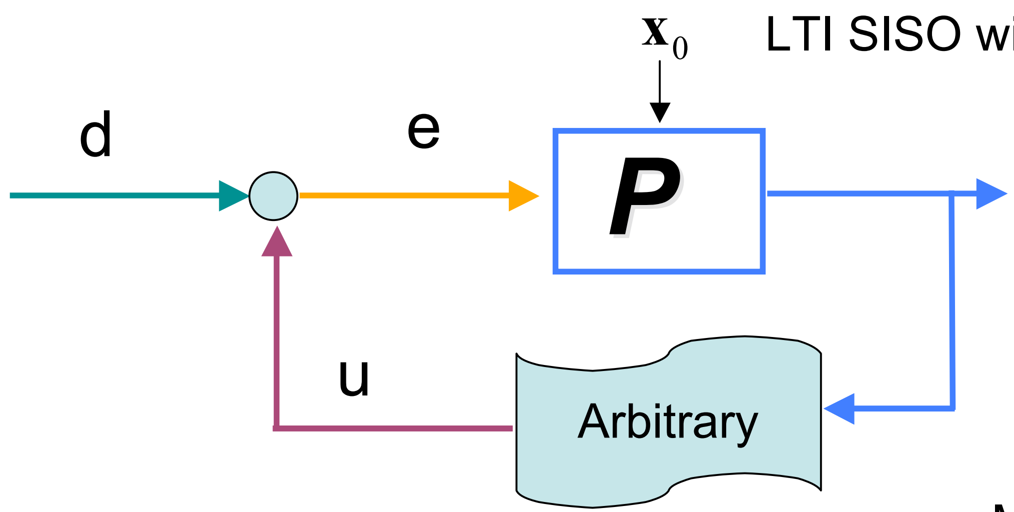
(Martins Ph.D. Thesis, 04)

$$h\left(\mathbf{e}^k\right) \geq h\left(\mathbf{d}^k\right) + I\left(\mathbf{x}_0;\mathbf{e}^k\right)$$

$$h_\infty(\mathbf{e}) \geq h_\infty(\mathbf{d}) + \liminf_{k\to\infty} \frac{I\left(\mathbf{x}_0;\mathbf{e}^k\right)}{k}$$

A conservation law using differential entropy

LTI SISO with random initial state



Mutual information

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$$h_\infty(\mathbf{e}) \geq h_\infty(\mathbf{d}) + \liminf_{k\to\infty} \frac{I\left(\mathbf{x}_0;\mathbf{e}^k\right)}{k}$$

Stability implies:

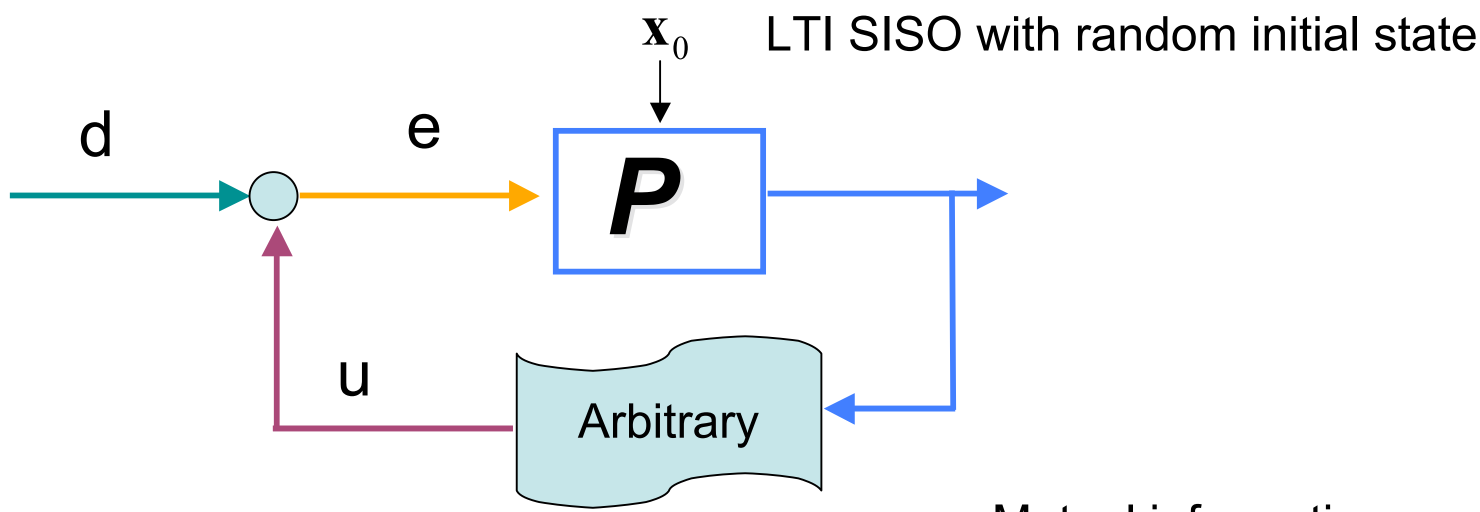
$$\liminf_{k\to\infty} \frac{I\left(\mathbf{x}_0;\mathbf{e}^k\right)}{k} \geq \sum_{unstable\ poles} \log\left|pole_i\right|$$

Baillieul

Tatikonda

Nair

A conservation law using differential entropy



Mutual information

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Stability

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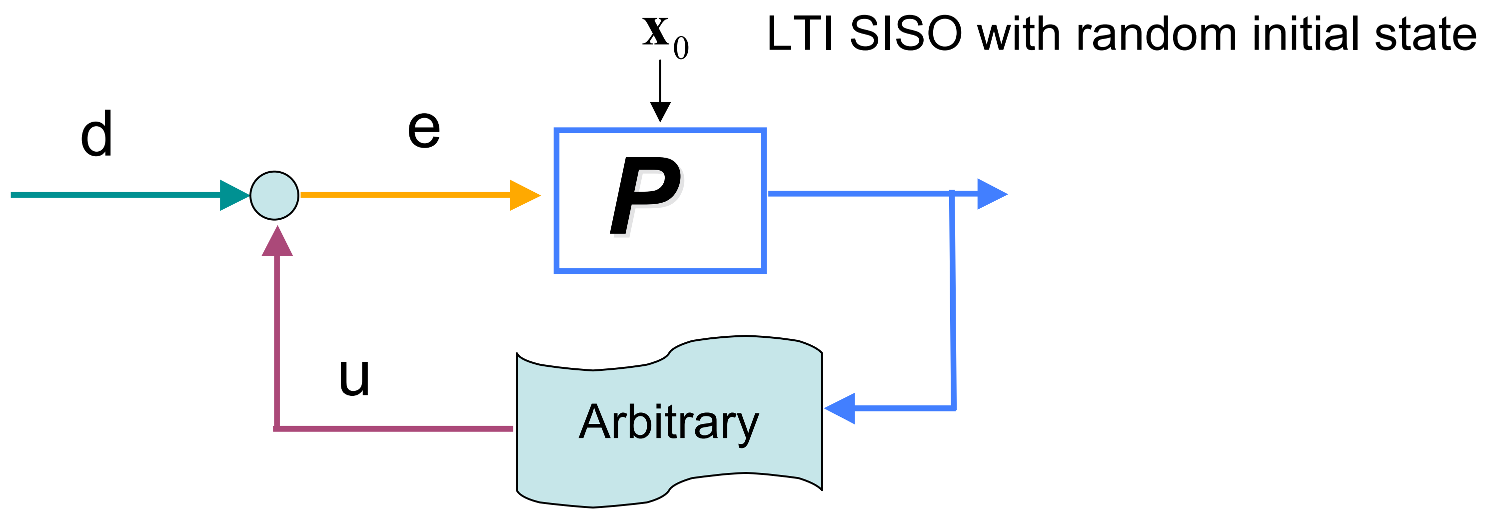


What can we do with this formula?

$$h_\infty(\mathbf{e}) \geq h_\infty(\mathbf{d}) + \sum_{unstable\ poles} \log|pole_i|$$

$$\mathbf{a}^k = \left(\mathbf{a}(0), \cdots, \mathbf{a}(k)\right)$$

$$h\left(\mathbf{a}^k\right) = -\int_{\Re^k} p_{a^k}(\gamma) \log_2 p_{a^k}(\gamma) d\gamma$$

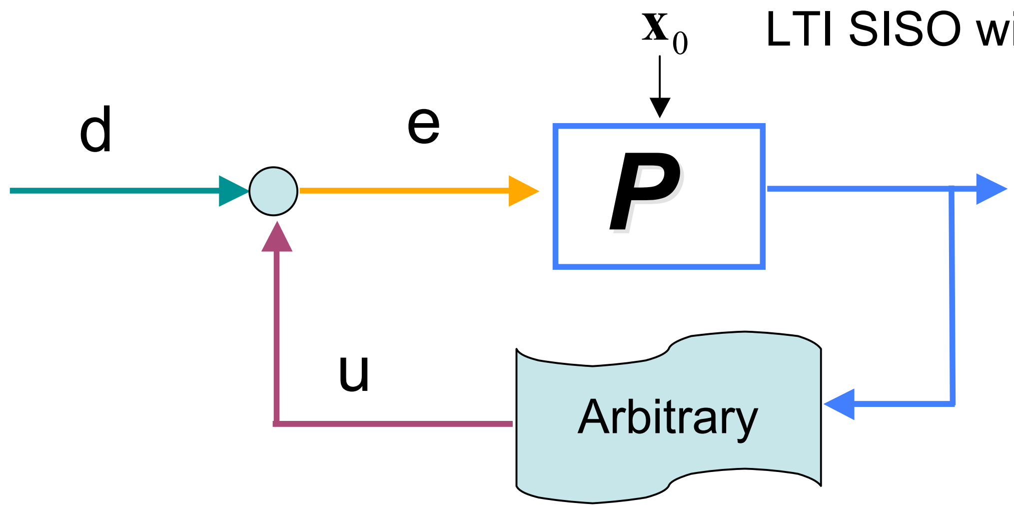
$$h\left(\mathbf{a}^k\right) \leq \frac{1}{2} \log\left((2\pi e)^k \left|\Sigma_{\mathbf{a}^k}\right|\right) \leq \frac{1}{2} \sum_{i=1}^{k} \log\left(2\pi e \sigma^2_{\mathbf{a}(i)}\right)$$

Equality is achieved if $\mathbf{a}^k$ is Gaussian

Equality is achieved if $\mathbf{a}(i)$ are uncorrelated

A conservation law using differential entropy: lower bound on the variance gain

LTI SISO with random initial state



$$h_\infty(\mathbf{e}) \geq h_\infty(\mathbf{d}) + \sum_{unstable\ poles} \log|pole_i|$$

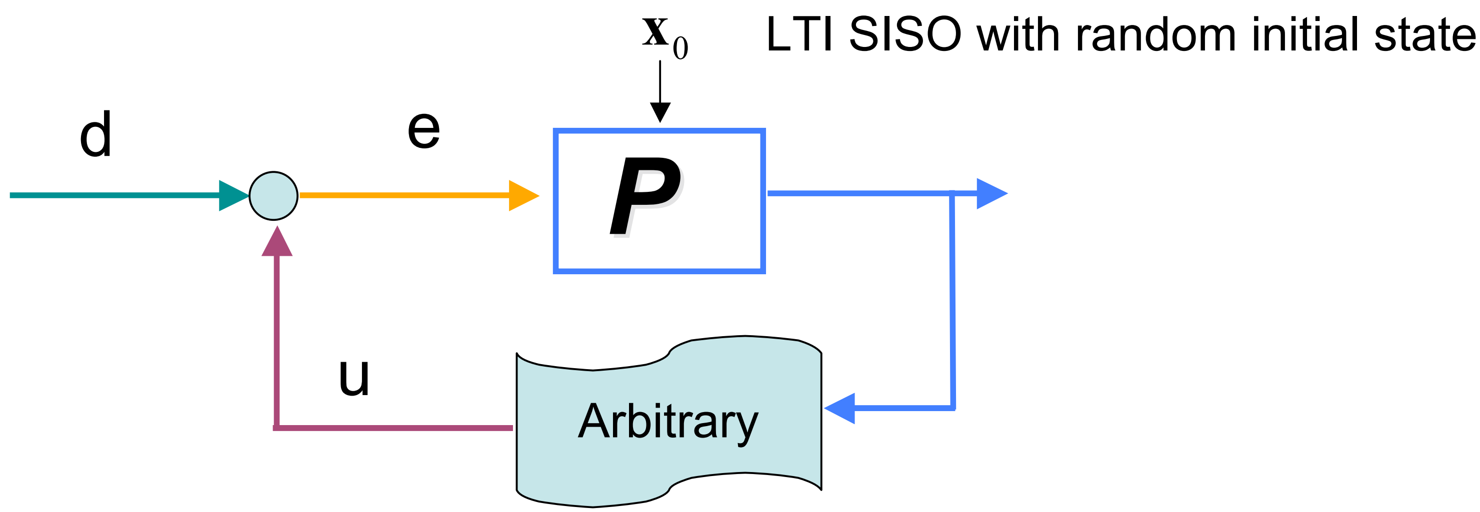
Choose **d** i.i.d. Gaussian with variance σ_d^2:

$$\limsup_{k\to\infty} \sigma_{e(k)}^2 \geq \sigma_d^2 \prod_{unstable\ poles} |pole_i|^2$$

Equality is achieved if is Gaussian

Equality is achieved if $\mathbf{a}(i)$ are uncorrelated

A conservation law using differential entropy: lower bound on the variance gain



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$$\limsup_{k\to\infty} \sigma_{u(k)}^2 \geq \sigma_d^2 \left(\prod_{unstable\ poles} |pole_i|^2 - 1\right)$$

$\mathbf{a}(i)$

Equality is achieved if is Gaussian

Equality is achieved if are uncorrelated

(Braslavsky, Middleton, Freudenberg 04)

Differential Entropy: power spectral bound

$$h_\infty(\mathbf{a}) = \limsup_{k\to\infty} \frac{h(\mathbf{a}\;)}{k} = \limsup_{k\to\infty} \frac{\sum_{i=1}}{k}$$

$$h(\mathbf{a}^k) = h(\mathbf{a}(k) \mid \mathbf{a}^{k-1}) + h(\mathbf{a}^{k-1})$$

$$\mathbf{a}^k = (\mathbf{a}(0), \ldots, \mathbf{a}(k))$$

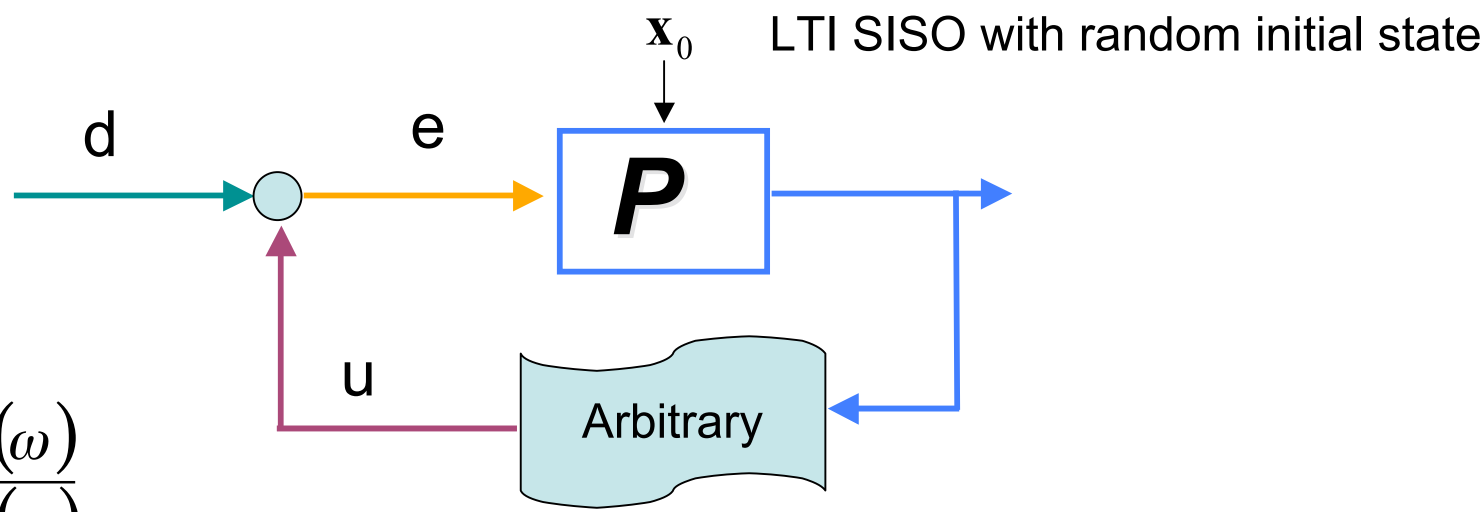
(Entropy rate)

Under Asymptotic stationarity:

$$h_\infty(\mathbf{a}) \le \frac{1}{4\pi} \int_{-\pi}^{\pi} \log(2\pi e F_a(\omega))\, d\omega$$

Equality is achieved if $\mathbf{a}$ is Gaussian

A conservation law using differential entropy: extension of Bode's integral formula



$$S_{e,d}^2(\omega) = \frac{F_e(\omega)}{F_d(\omega)}$$

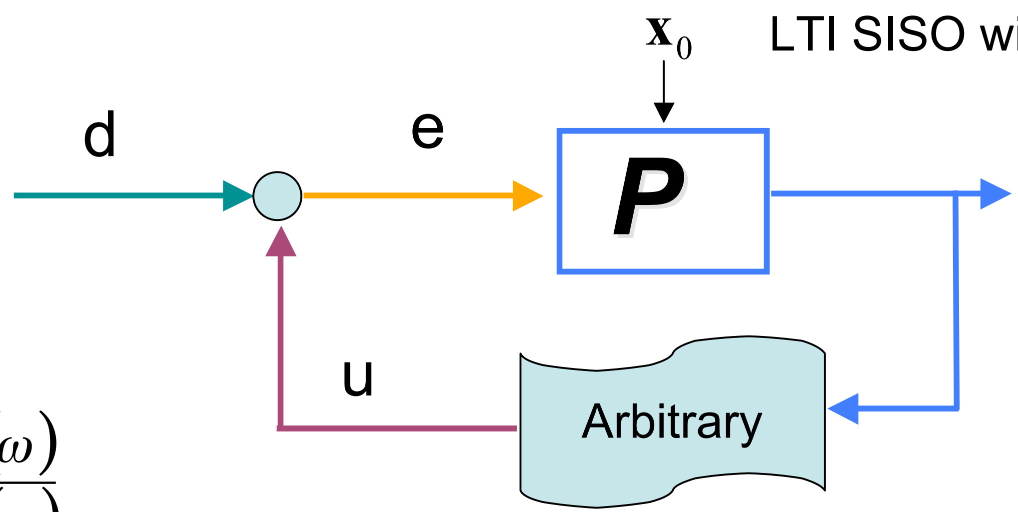
$$h_\infty(\mathbf{e}) \geq h_\infty(\mathbf{d}) + \sum_{unstable\ poles} \log|pole_i|$$

Choose **d** Gaussian, wide-sense asympt. stationary:

$$\int_{-\pi}^{\pi} \log(2\pi e F_e(\omega))d\omega \geq \int_{-\pi}^{\pi} \log(2\pi e F_d(\omega))d\omega + 4\pi \sum_{unstable\ poles} \log|pole_i|$$

A conservation law using differential entropy: extension of Bode's integral formula

LTI SISO with random initial state



$$S_{e,d}^2(\omega) = \frac{F_e(\omega)}{F_d(\omega)}$$

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$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log S_{e,d}^2(\omega)d\omega \geq 2 \sum_{unstable\ poles} \log|pole_i|$$

(Bode Integral formula!)

$$\mathbf{a}^k = \left(\mathbf{a}(0), \cdots, \mathbf{a}(k)\right)$$

$$h\left(\mathbf{a}^k\right) = -\int_{\Re^k} p_{a^k}(\gamma) \log_2 p_{a^k}(\gamma) d\gamma$$

Upper-bound based

$$h\left(\mathbf{a}^k\right) \leq k \log\left(\overline{a}\right)$$

$$\overline{a} \overset{def}{=} \inf\left\{x \in \Re_+ \mid P\left(\left|\mathbf{a}(i)\right| > x\right) = 0, i \in \{1, \ldots, k\}\right\}$$

A few bounds:

$$\mathbf{a}^k = \left(\mathbf{a}(0), \cdots, \mathbf{a}(k)\right)$$

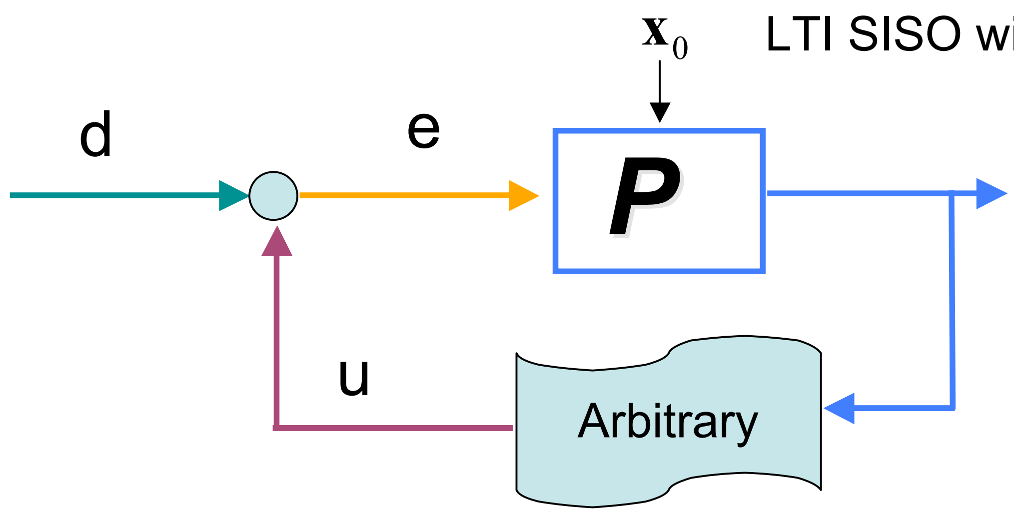
$$h\left(\mathbf{a}^k\right) = -\int_{\Re^k} p_{a^k}(\gamma) \log_2 p_{a^k}(\gamma)\, d\gamma$$

Upper-bound based

$$h\left(\mathbf{a}^k\right) \leq k \log\left(\overline{a}\right)$$

Achieved if a^k is uniformly distributed

A Conservation Law



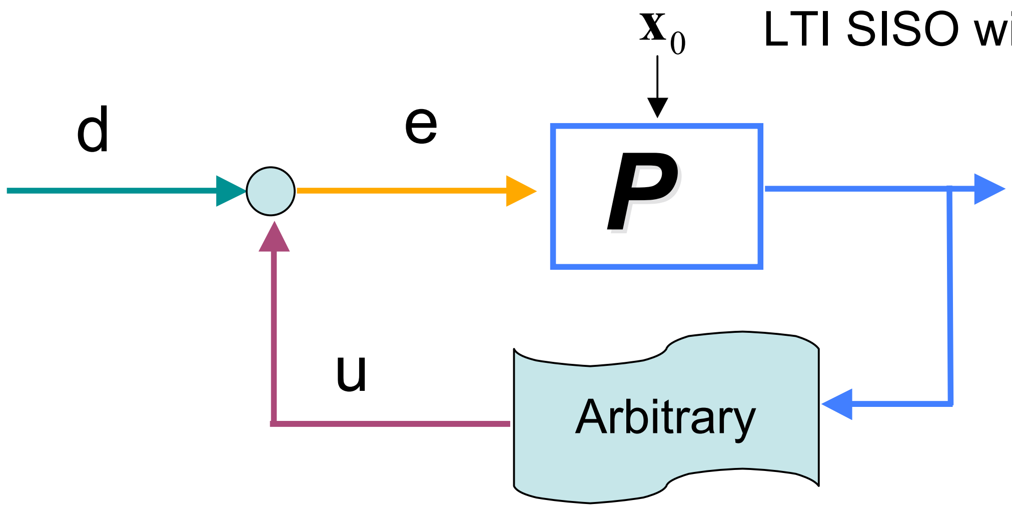
$$h_\infty(\mathbf{e}) \geq h_\infty(\mathbf{d}) + \sum_{|\lambda_i(A)|>1} \log|\lambda_i(A)|$$

Choose **d** i.i.d. uniformy distributed between $-\bar{d}$ and $\bar{d}$:

$$\Rightarrow \limsup_{k\to\infty} \bar{e}(k) \geq \bar{d} \prod_{unstable\ poles} |pole_i|$$

A Conservation Law

LTI SISO with random initial state



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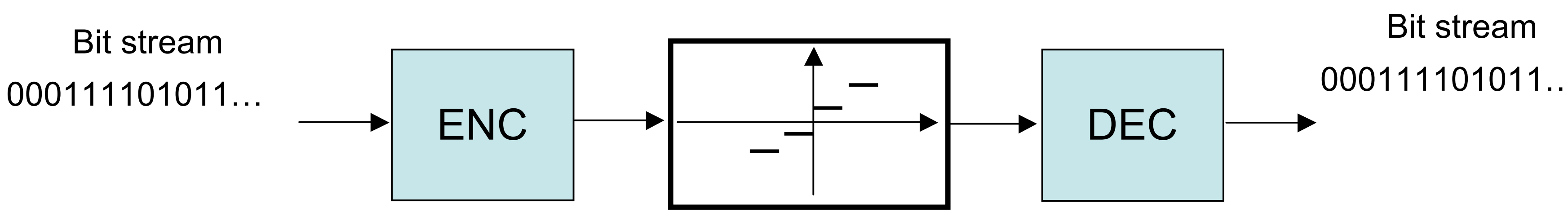
$$\Rightarrow \quad \limsup_{k\to\infty} \bar{e}(k) \geq \bar{d} \prod_{unstable\ poles} |pole_i|$$

$$\limsup_{k\to\infty} \bar{u}(k) \geq \bar{d}\left(\prod_{unstable\ poles} |pole_i| - 1\right)$$

Limits in the presence of finite capacity feedback

Information capacity is the supremum of the bit-rate for which information can be transmitted through a medium:

Examples:



quantizer
with *M* levels

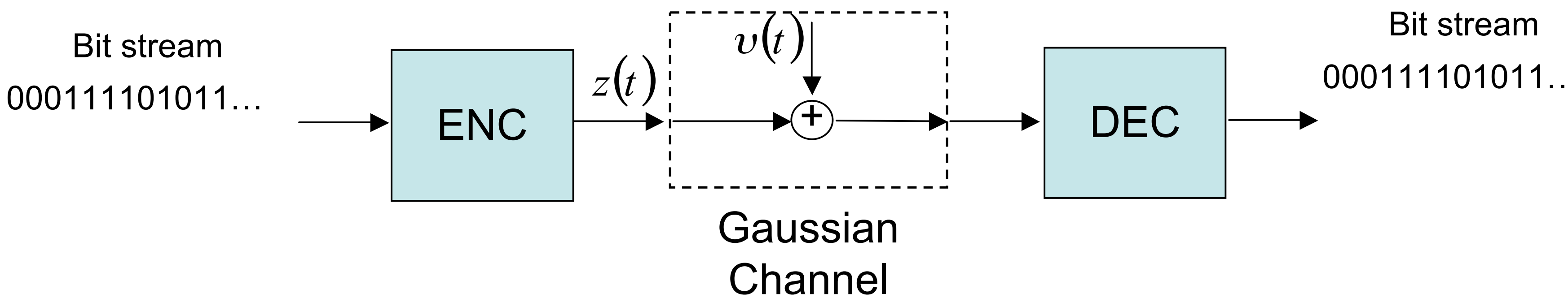
If $N(t)$ represents the total number of bits transmitted up to time *t* then we know that

$$\sup_t \frac{N(t)}{t} \leq \log_2 M$$

← Capacity

Information capacity is the supremum of the bit-rate for which information can be transmitted through a medium:

Examples:

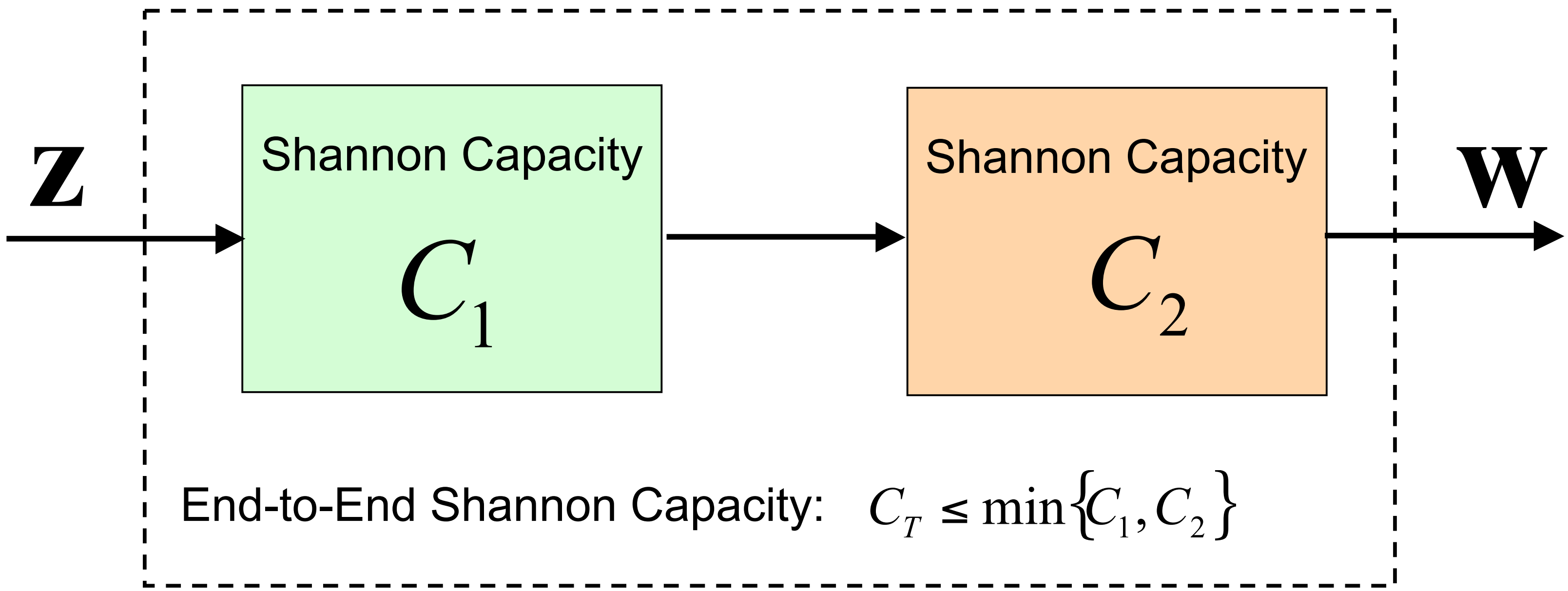


If $N(t)$ represents the total number of bits transmitted up to time *t* then we know that

$$\begin{cases} \sup_t \dfrac{N(t)}{t} \leq \dfrac{1}{2}\log_2\left(1+\dfrac{\sigma_z^2}{\sigma_v^2}\right) \\ P[Error(t)] \rightarrow 0 \end{cases}$$

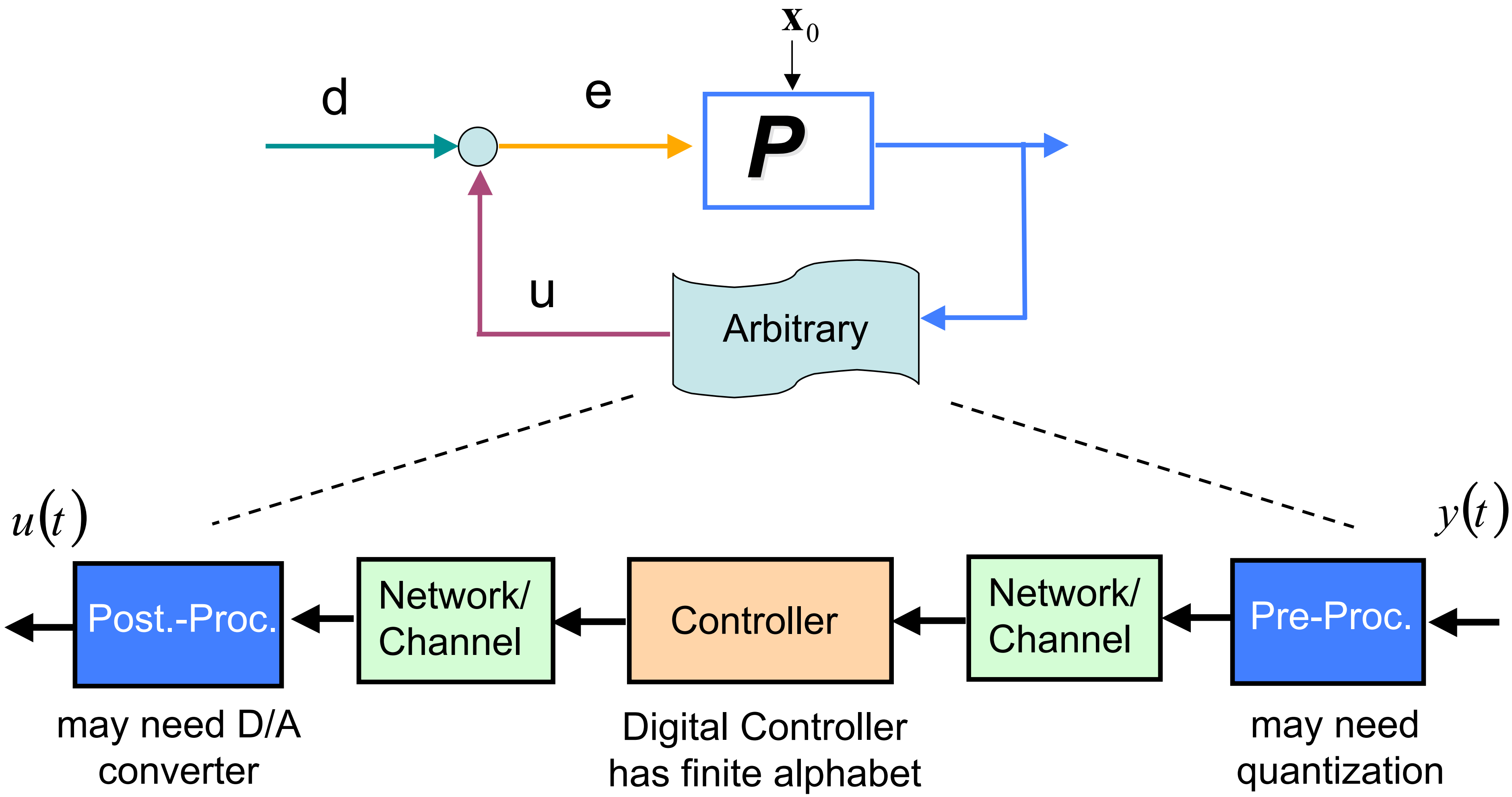
← Shannon Capacity

Data-processing inequality



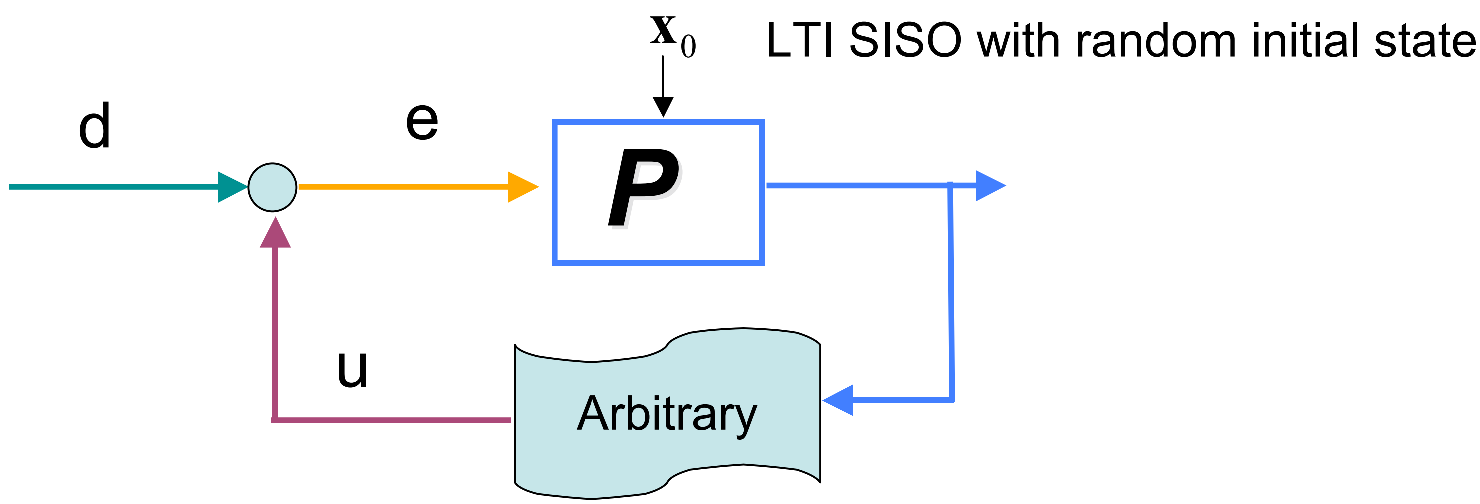
$$I(\mathbf{z}, \mathbf{w}) \leq C_T$$

Limits in the presence of finite capacity feedback



$C_f < \infty$ Finite Capacity Feedback

Limits in the presence of finite capacity feedback



$$C_f < \infty$$

Finite capacity feedback has impact on disturbance attenuation:

New Bound $$\frac{1}{2\pi}\int_{-\pi}^{\pi}\min\{\log(S_{e,d}(\omega)),0\}d\omega \geq \sum_{unstable\ poles}\log|pole_i| - C_f$$

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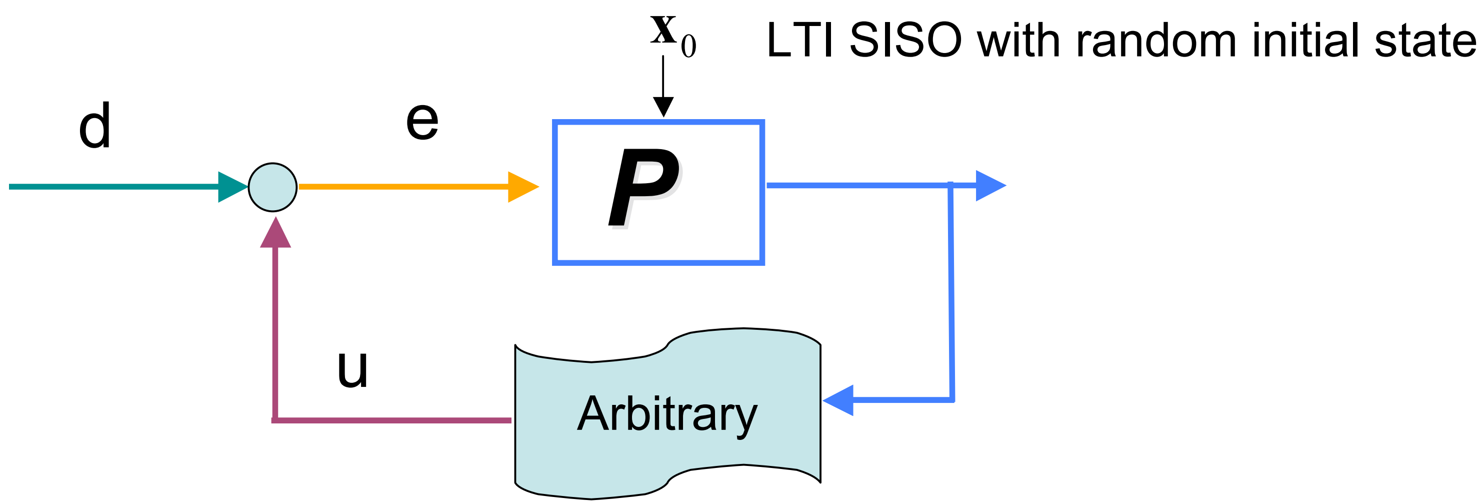

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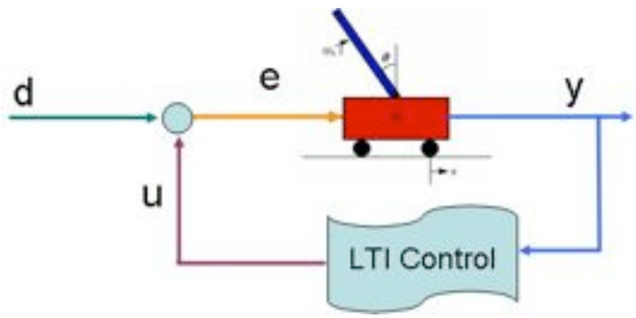
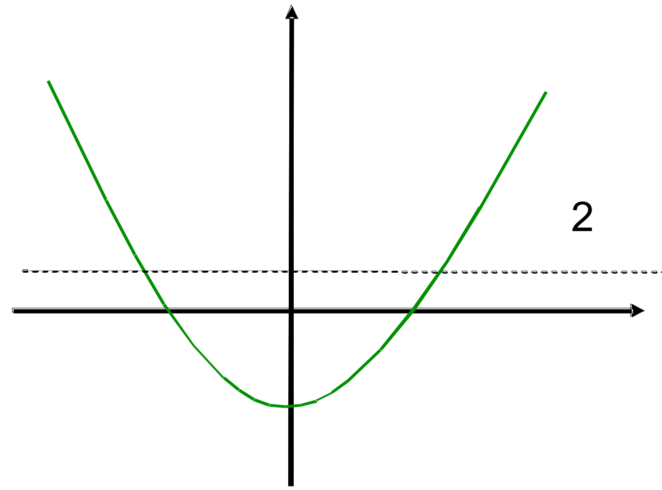
Original Bode formula resulting from Causality

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log(S_{e,d}(\omega))\, d\omega \geq \sum_{unstable\ poles} \log|pole_i|$$

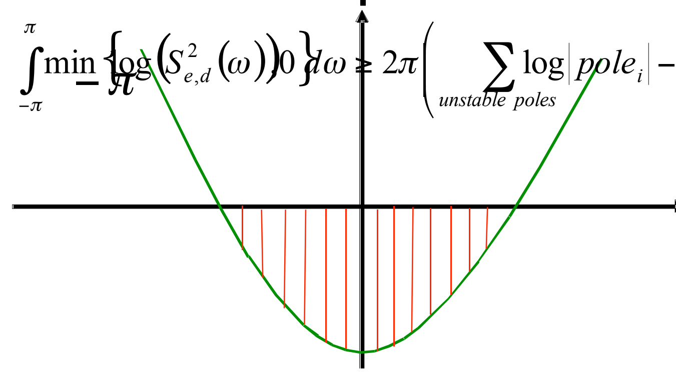
Limits in the presence of finite capacity feedback

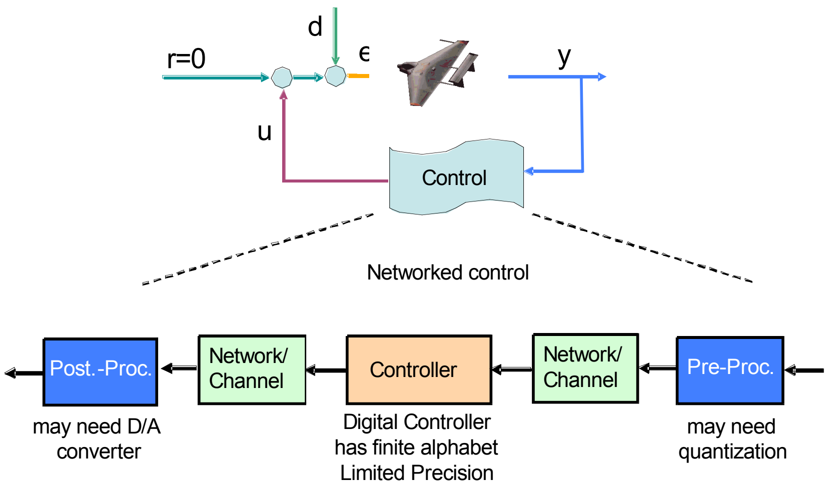


$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log S_{e,d}^2(\omega)\,d\omega \ge 2\sum_{\text{unstable poles}}\log|pole_i|$$



$$\int_{-\pi}^{\pi}\min\{\log(S_{e,d}^2(\omega)),0\}\,d\omega \ge 2\pi\left(\sum_{\text{unstable poles}}\log|pole_i| - C_f\right)$$





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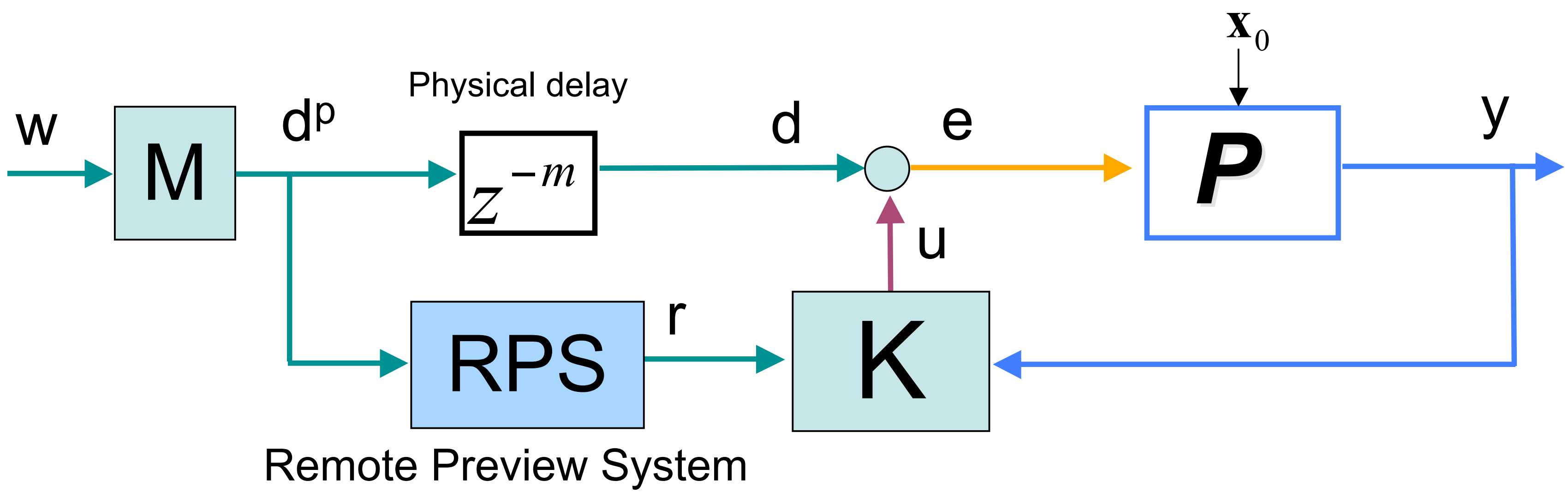
New Bound

Limits in the presence of a finite capacity preview

Motivation for Using Early Warning Information in a Feedback Loop

Feedback in the Presence of a Remote Preview



Motivation for Using Early Warning Information in a Feedback Loop

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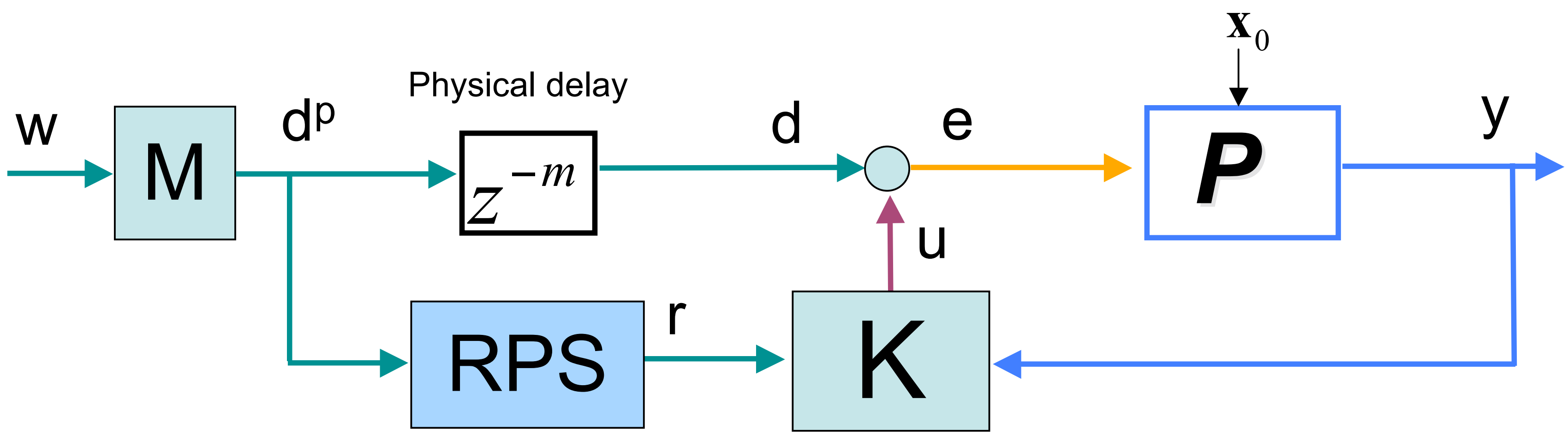

Remote Preview System

Can we beat the standard (no RPS) limitation?:

$$S_{e,d}^2(\omega) = \frac{F_e(\omega)}{F_d(\omega)}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log S_{e,d}^2(\omega)\,d\omega \geq 2\sum_{unstable\ poles}\log|pole_i|$$

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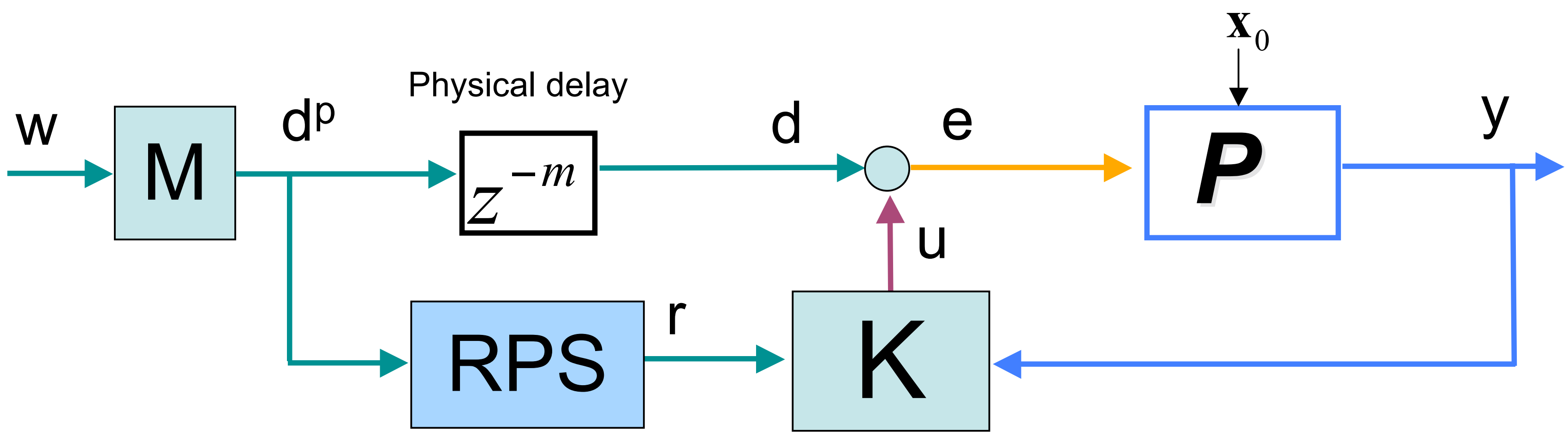

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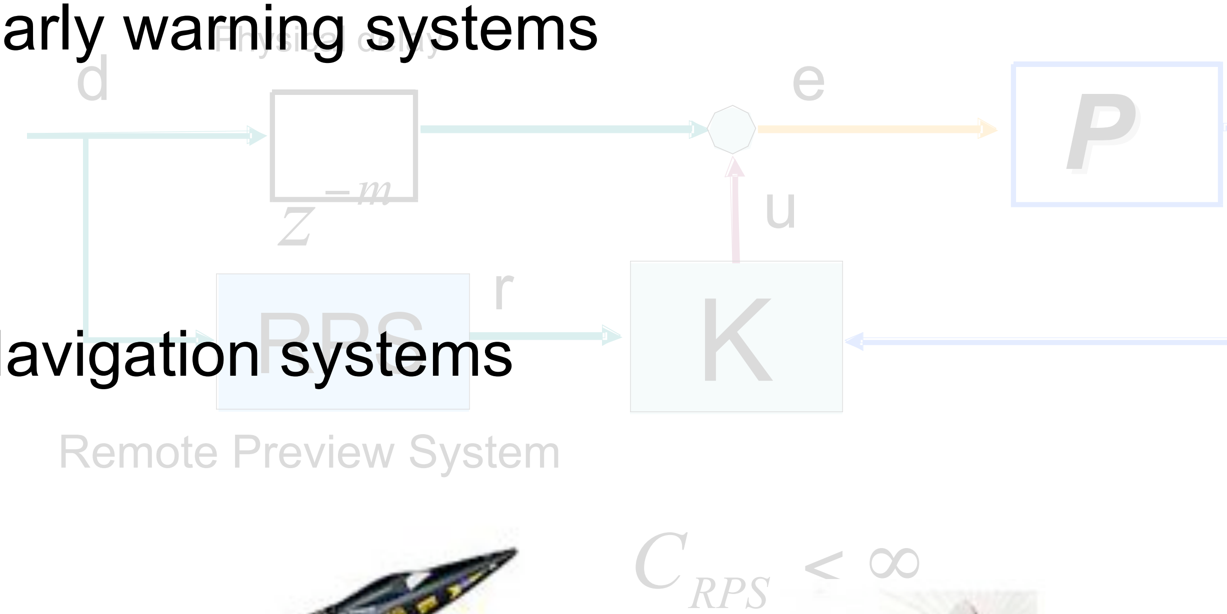
What can we do, subject to the following constraint?

$$I_\infty(\mathbf{d}, \mathbf{r}) \leq C$$

← Capacity of the comm. link

Early warning systems

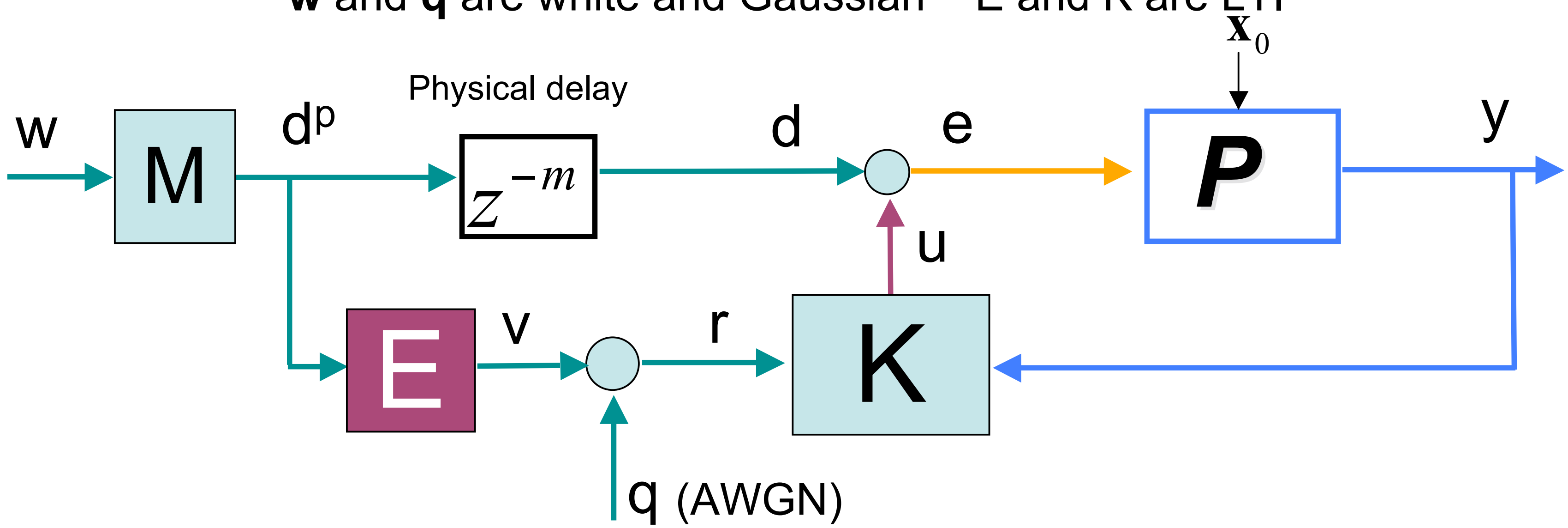
Navigation systems



Motivation for Using Early Warning Information in a Feedback Loop

The Linear, Time-Invariant and Gaussian Case

w and **q** are white and Gaussian   E and K are LTI



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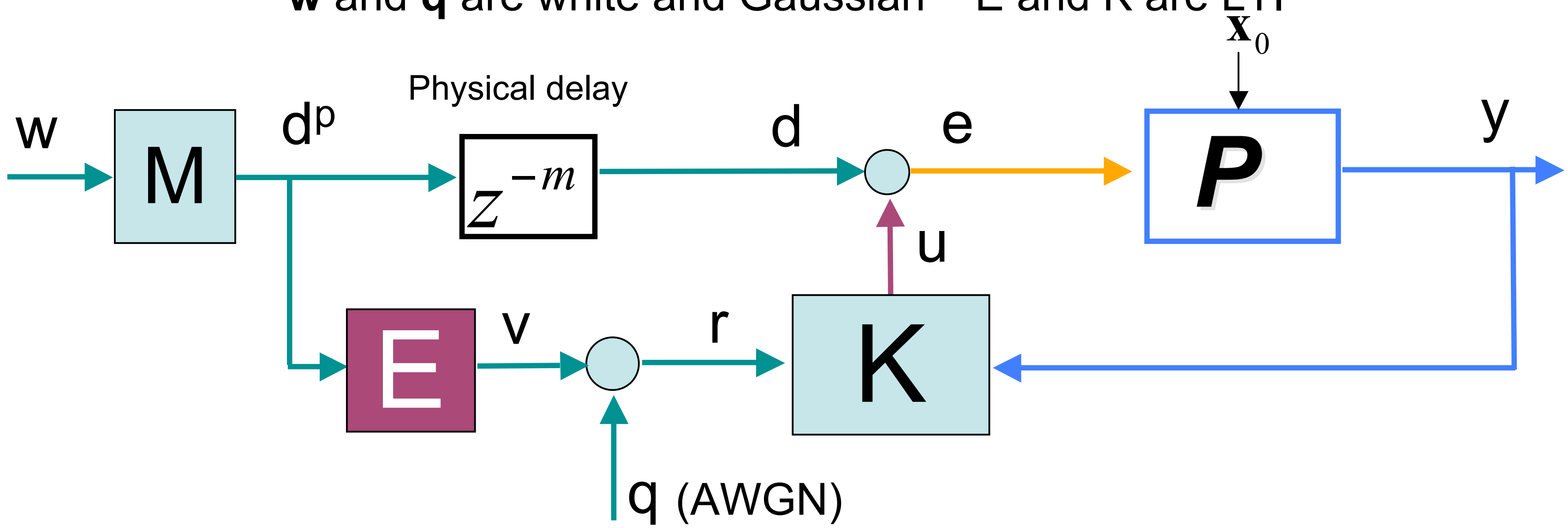

Can we beat the standard (no Channel) limitation?:

$$S_{e,d}^2(\omega) = \frac{F_e(\omega)}{F_d(\omega)} \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} \log S_{e,d}^2(\omega)\, d\omega \geq 2 \sum_{unstable\ poles} \log |pole_i|$$

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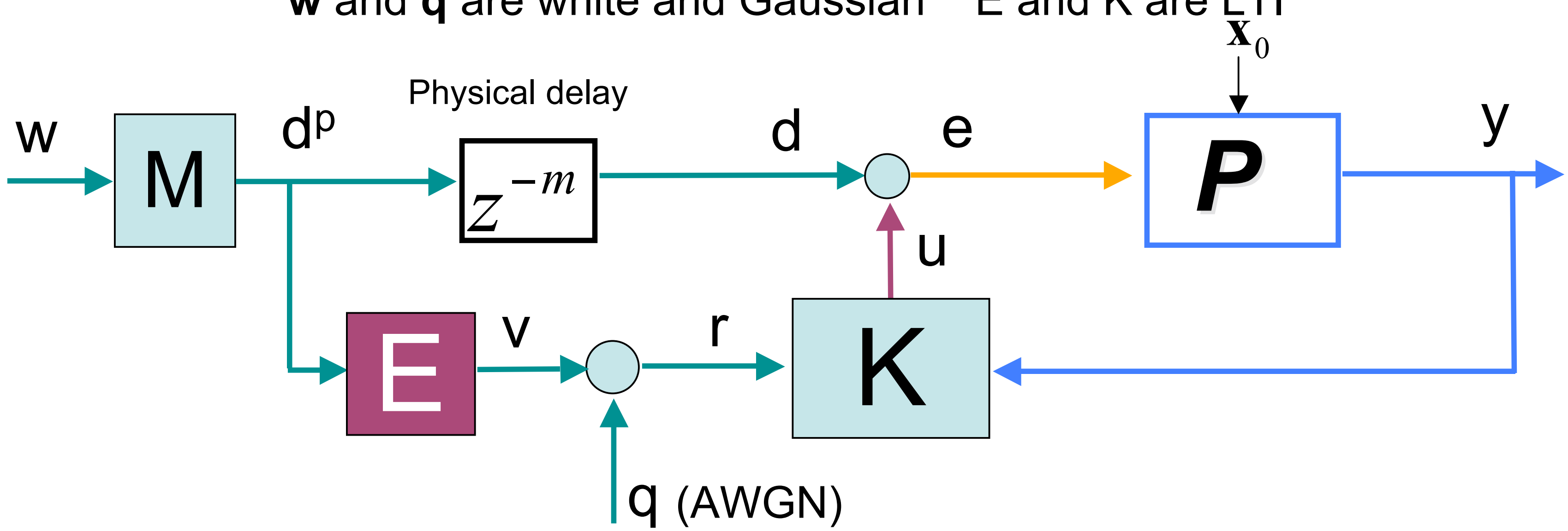

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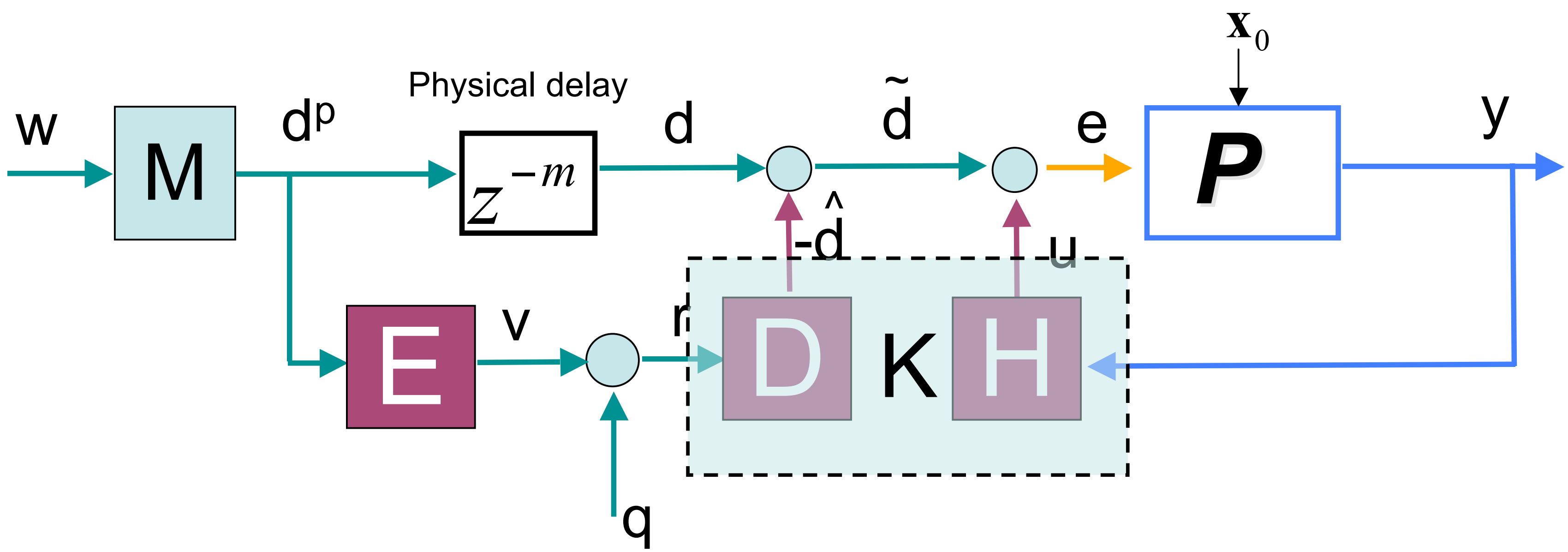
What can we do with the following power constraint?

$$\sup_k Var(\mathbf{v}(k)) \leq 1$$

Motivation for Using Early Warning Information in a Feedback Loop

The Linear, Time-Invariant and Gaussian Case



E, D and H are LTI

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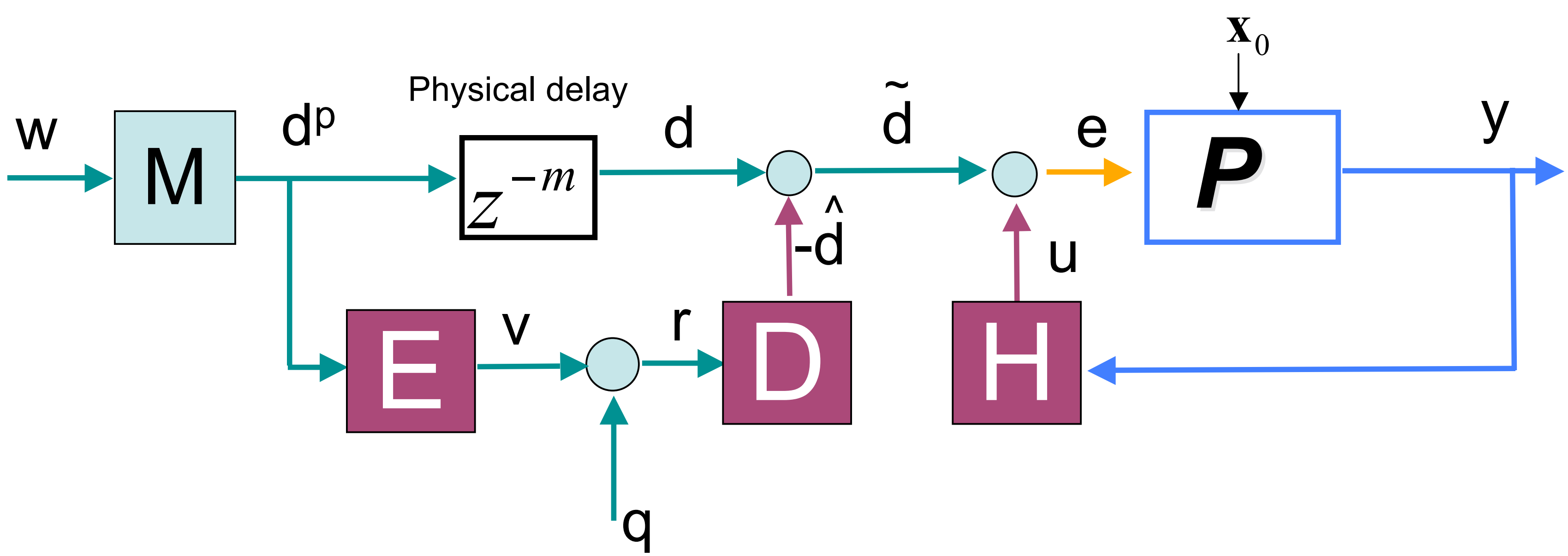

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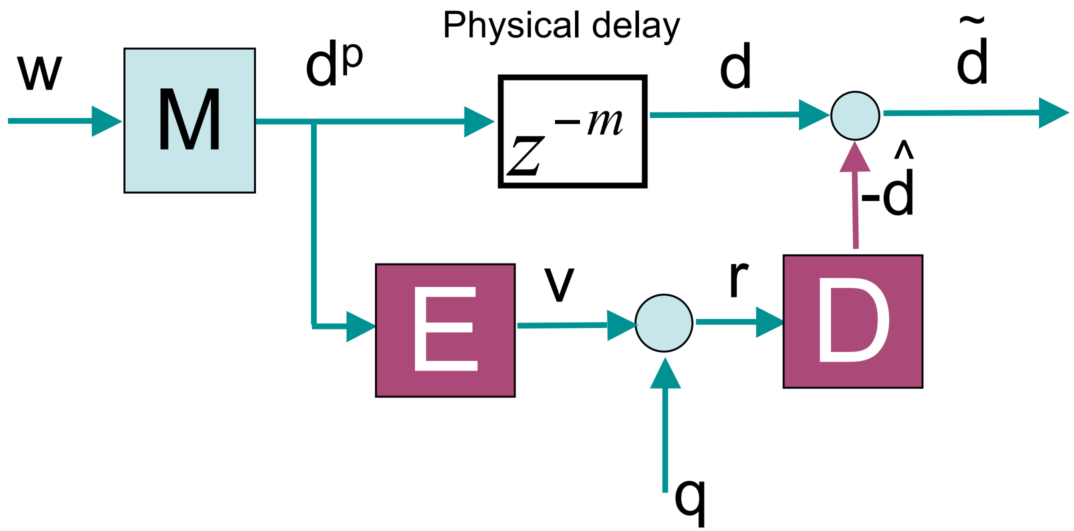
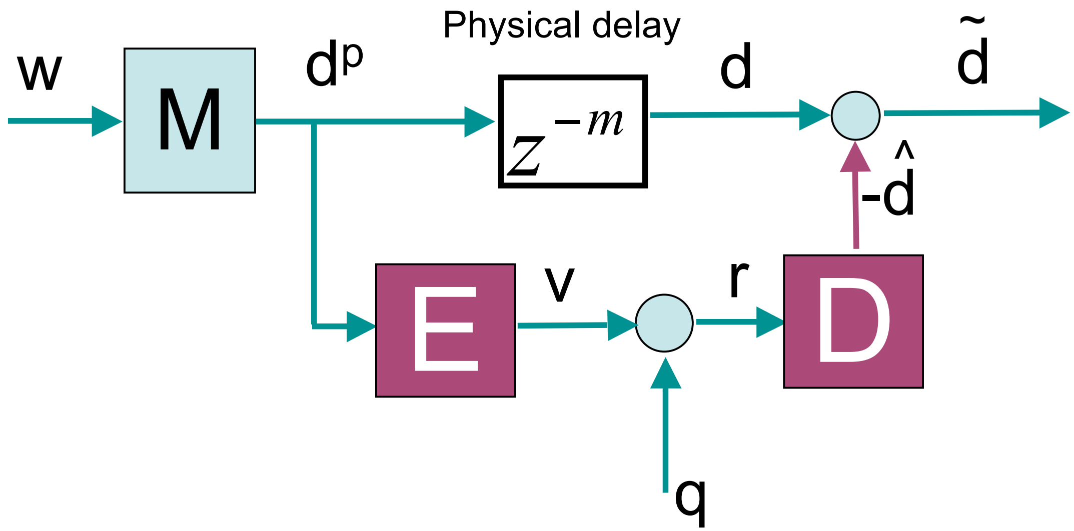
Motivation for Using Early Warning Information in a Feedback Loop

Forward Loop:



Motivation for Using Early Warning Information in a Feedback Loop

Forward Loop:



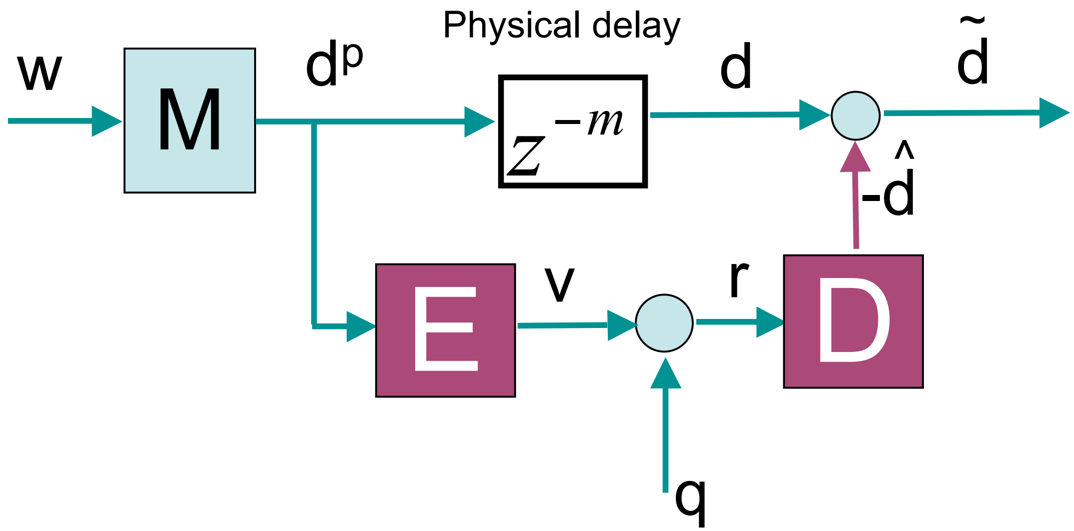
AD HOC choice:

$$E = M^{-1}$$

Send the innovations through the channel

Motivation for Using Early Warning Information in a Feedback Loop

Forward Loop:



AD HOC choice:

$$E = M^{-1}$$

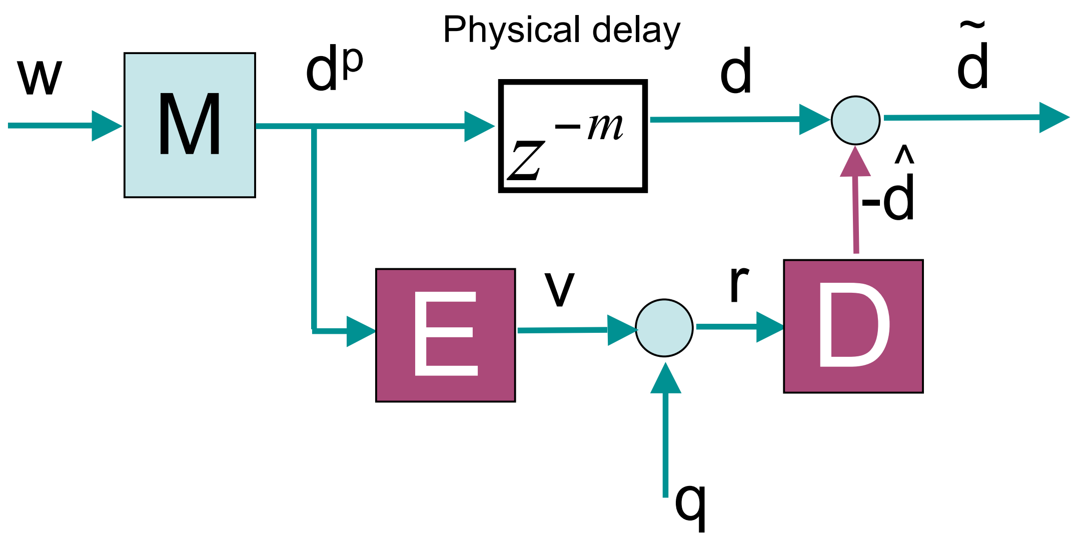
Send the innovations through the channel

$$D = -z^{-m}\left(\frac{1}{1+Var(q)}\right)M$$

Optimal estimator

Motivation for Using Early Warning Information in a Feedback Loop

Forward Loop:



Computing Power Spectral Densities, leads to:

$$\log \sqrt{\frac{F_{\tilde{d}}(\omega)}{F_d(\omega)}} = \frac{1}{2} \log \left(\frac{1}{1 + \frac{1}{Var(q)}} \right) = -C$$

Motivation for Using Early Warning Information in a Feedback Loop

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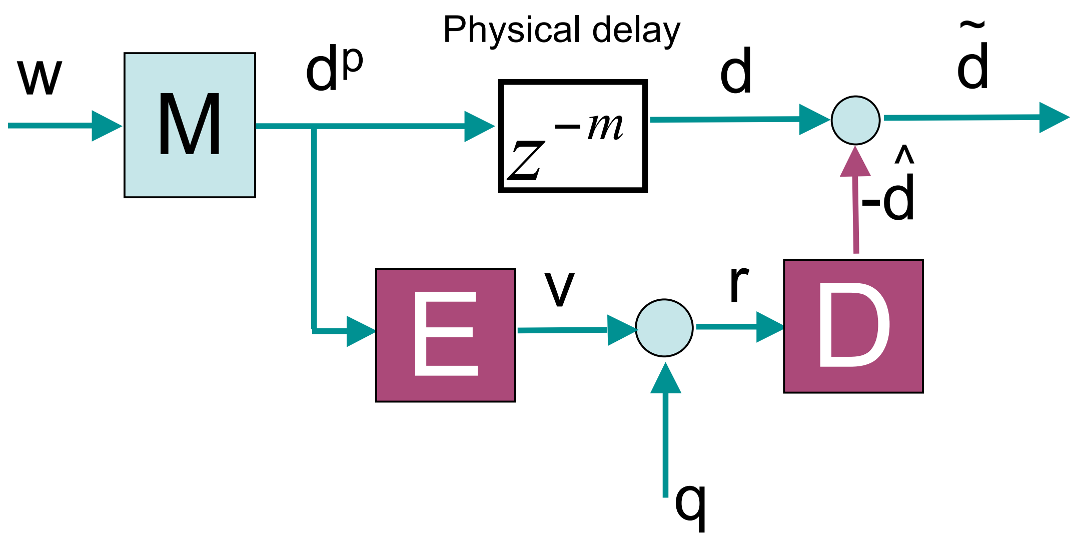

Could we have done Better ?

Computing Power Spectral Densities, leads to:

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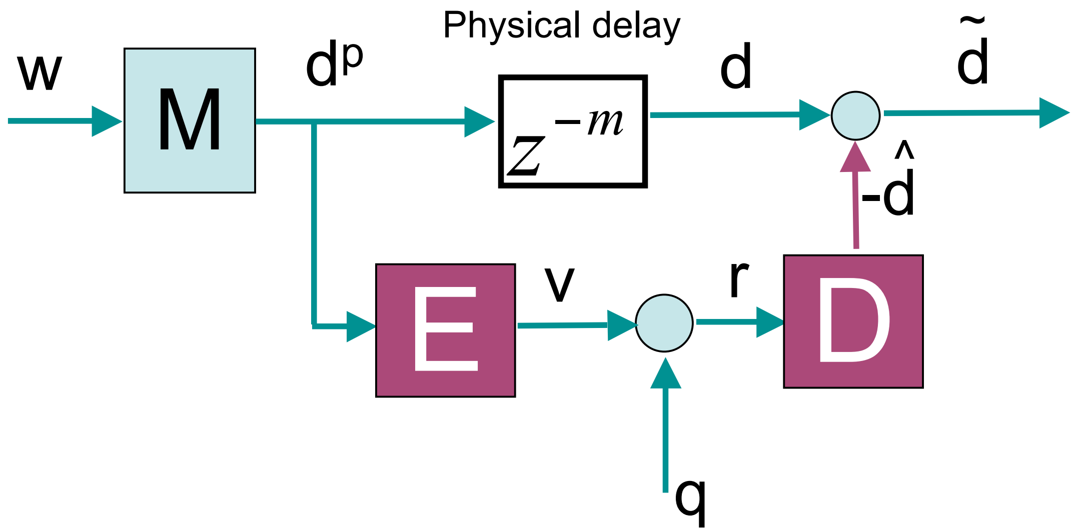
Voila!

Motivation for Using Early Warning Information in a Feedback Loop

Forward Loop:



Could we have done Better ?

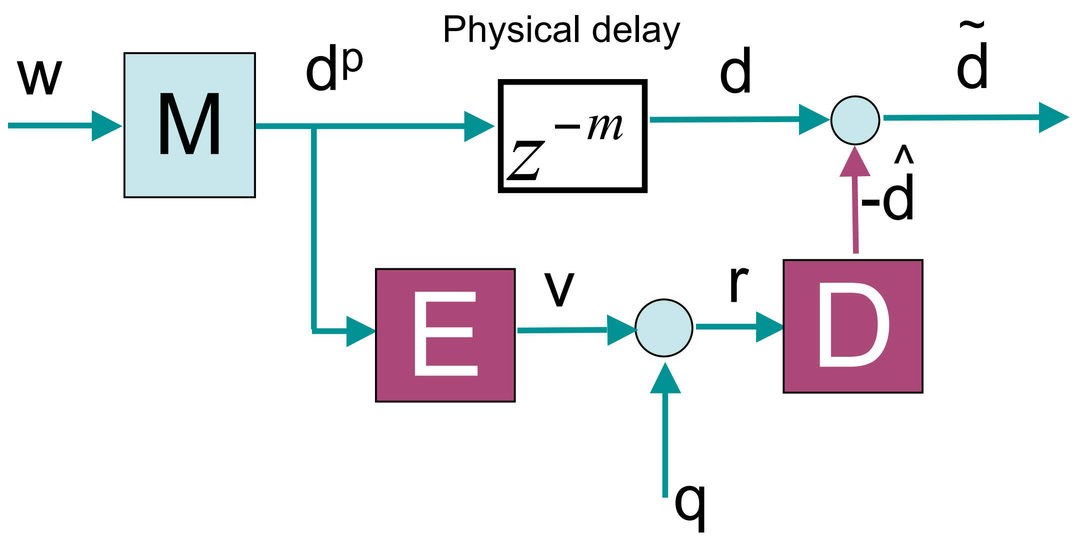
No!

Computing Power Spectral Densities, leads to:

$$\log\sqrt{\frac{F_{\tilde{d}}(\omega)}{F_d(\omega)}} = \frac{1}{2}\log\left(\frac{1}{1+\frac{1}{Var(q)}}\right) = -C \quad \Rightarrow \quad \frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left(\sqrt{\frac{F_{\tilde{d}}(\omega)}{F_d(\omega)}}\right)d\omega = -C$$

Motivation for Using Early Warning Information in a Feedback Loop

Forward Loop (General Case):



$$h_\infty(\tilde{\mathbf{d}}) \geq h_\infty(\mathbf{d}) - C$$

Motivation for Using Early Warning Information in a Feedback Loop

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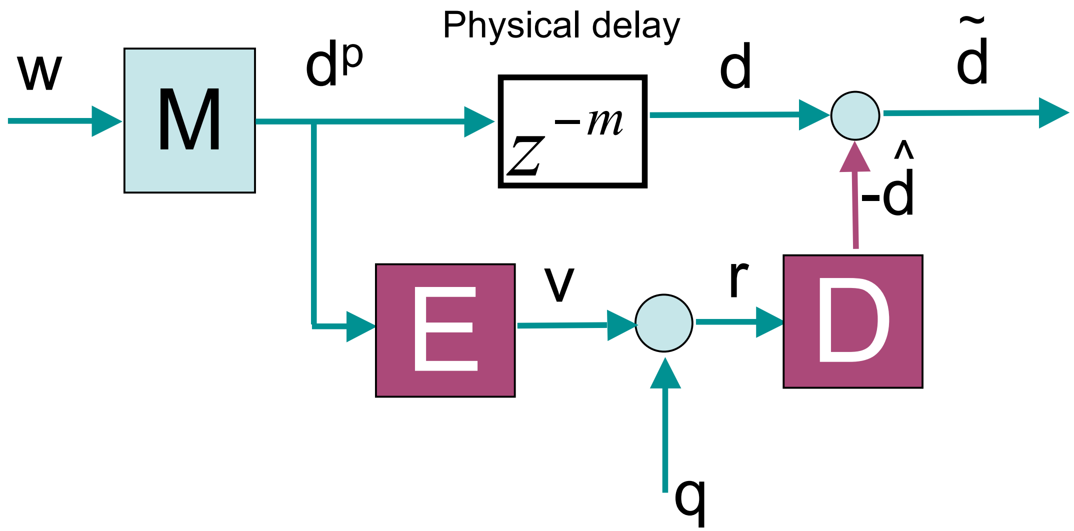

$$h_\infty(\tilde{\mathbf{d}}) \geq h_\infty(\mathbf{d}) - C \;\Rightarrow\; \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left(\sqrt{\frac{F_{\tilde{d}}(\omega)}{F_d(\omega)}}\right) d\omega \geq -C$$

Motivation for Using Early Warning Information in a Feedback Loop

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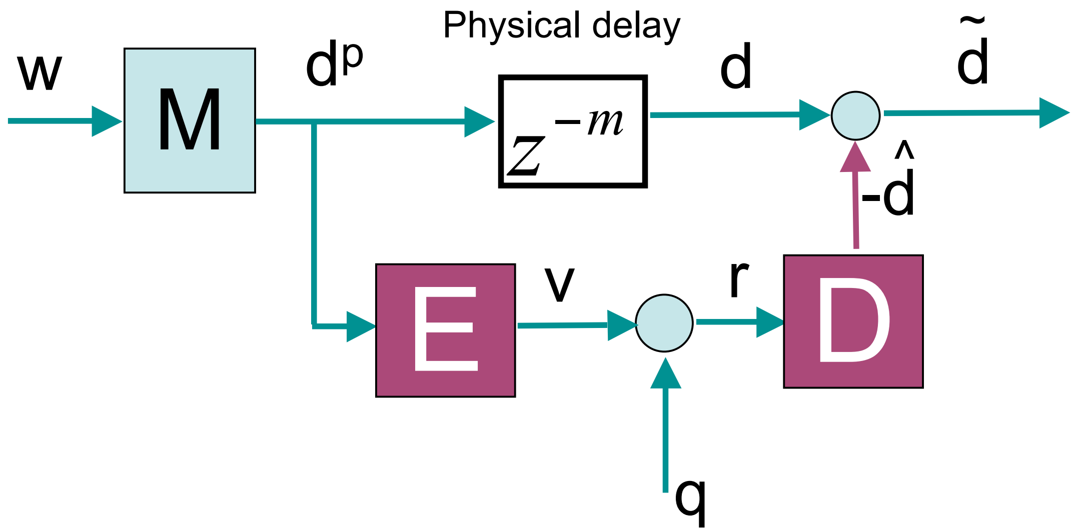


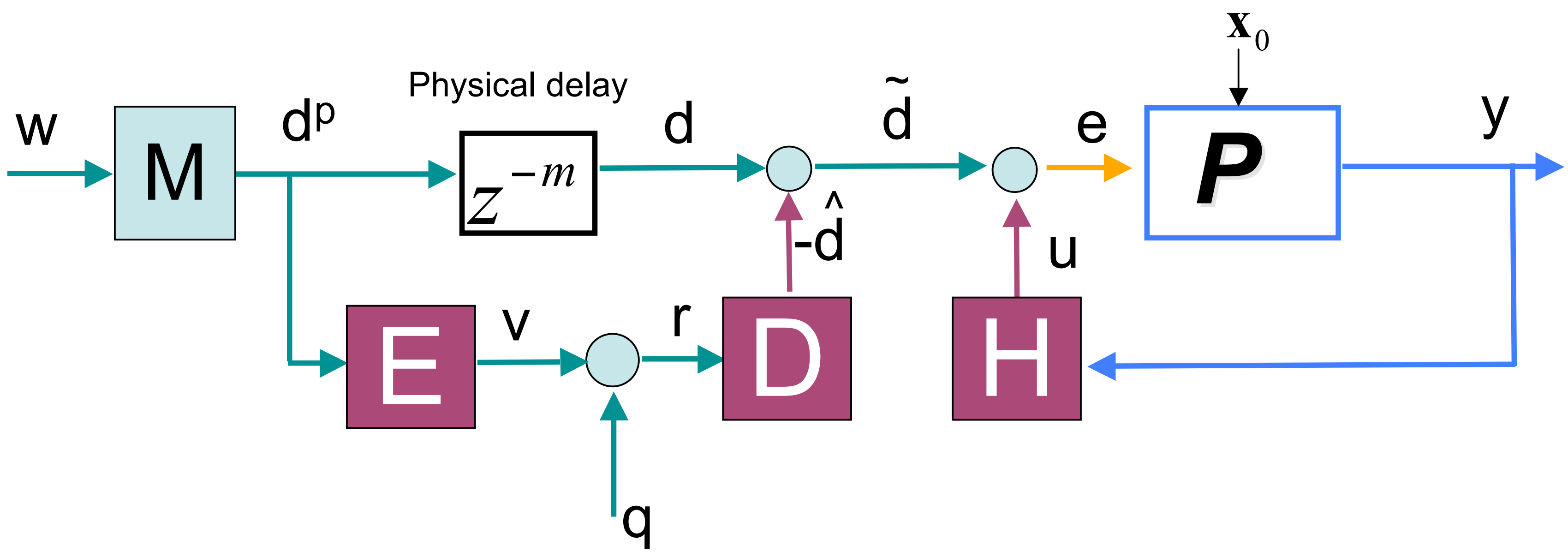
$$h_\infty(\tilde{\mathbf{d}}) \geq h_\infty(\mathbf{d}) - C \quad \Rightarrow \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left(\sqrt{\frac{F_{\tilde{d}}(\omega)}{F_d(\omega)}}\right) d\omega \geq -C$$

Proof:

$$h_\infty(\hat{\mathbf{d}} - \mathbf{d}) \geq h_\infty(\hat{\mathbf{d}} - \mathbf{d} \mid \hat{\mathbf{d}}) = h_\infty(\mathbf{d} \mid \hat{\mathbf{d}}) \geq h_\infty(\mathbf{d}) - I_\infty(\hat{\mathbf{d}}, \mathbf{d}) \geq h_\infty(\mathbf{d}) - I_\infty(\mathbf{v}, \mathbf{r}) \geq h_\infty(\mathbf{d}) - C$$

Motivation for Using Early Warning Information in a Feedback Loop

Complete Scheme with Linear Controller



$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left(\sqrt{\frac{F_{\tilde{d}}(\omega)}{F_d(\omega)}}\right) \geq -C$$

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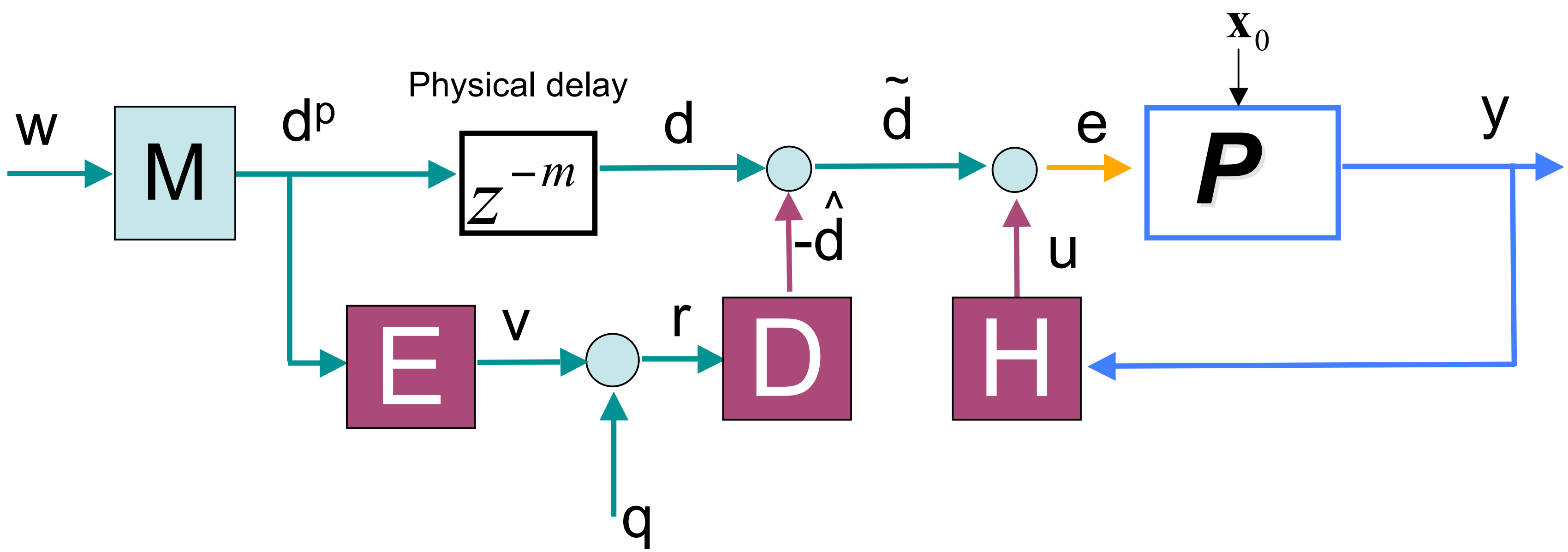
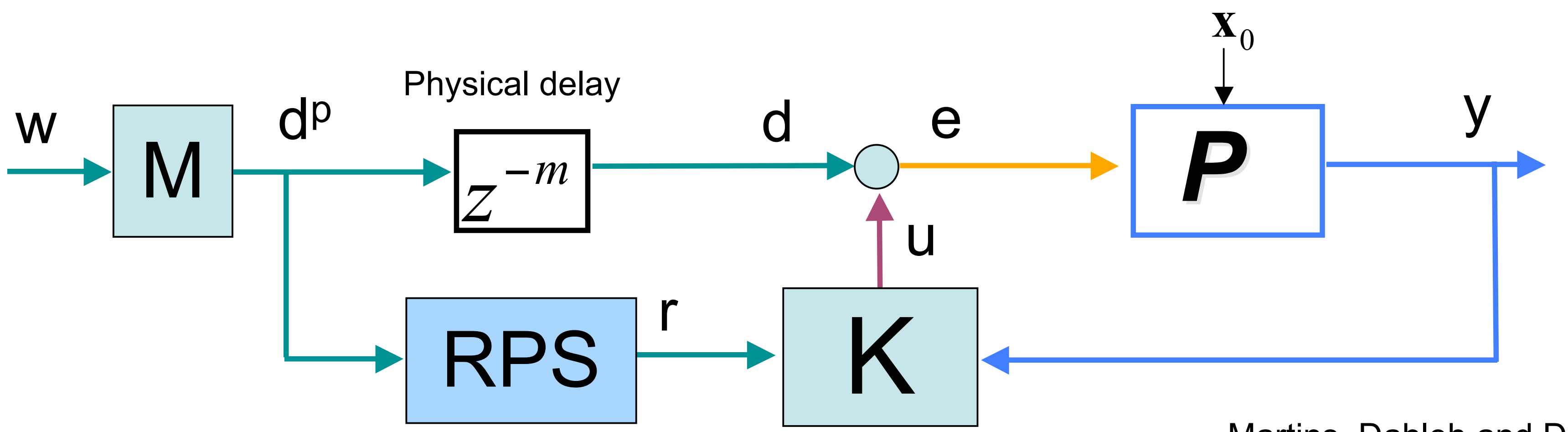


$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left(\sqrt{\frac{F_{\tilde{d}}(\omega)}{F_d(\omega)}}\right) \geq -C \quad \Rightarrow \quad \frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left(\sqrt{\frac{F_e(\omega)}{F_d(\omega)}}\right) \geq \sum_{unstable\ poles}\log|pole_i| - C$$

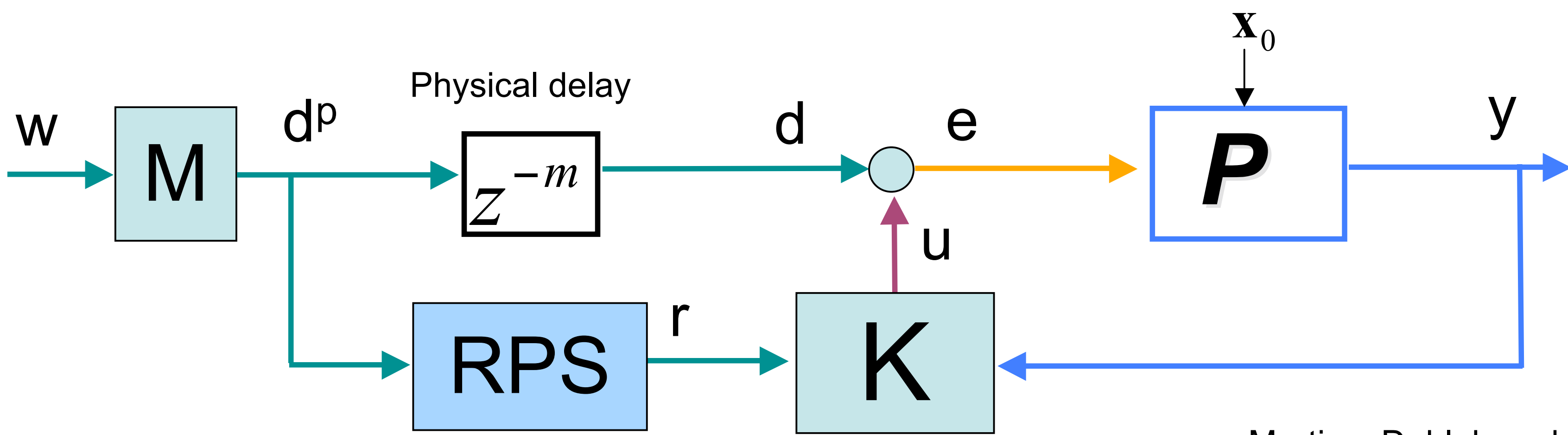
Complete Scheme (General Case)



Martins, Dahleh and Doyle
CDC 2005

$$h_\infty(\mathbf{e}) - h_\infty(\mathbf{d}) \geq I_\infty(\mathbf{x}(0), \mathbf{e}) - I_\infty(\mathbf{r}, \mathbf{d})$$

Complete Scheme (General Case)



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$$h_\infty(\mathbf{e}) - h_\infty(\mathbf{d}) \geq I_\infty(\mathbf{x}(0),\mathbf{e}) - I_\infty(\mathbf{r},\mathbf{d}) \leftarrow \leq C$$

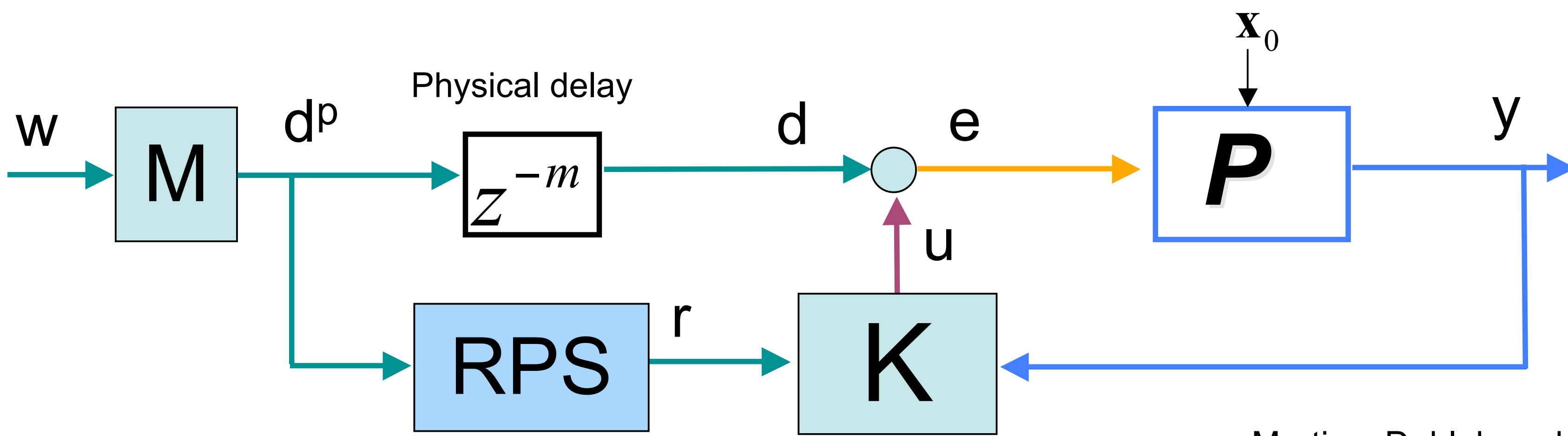
$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left(S_{e,d}(\omega)\right) \geq$$

$$\geq \sum_{\text{unstable poles}} \log|pole_i|$$

$$S(\omega) \overset{def}{=} \sqrt{\frac{S_e(\omega)}{S_d(\omega)}}$$

General Performance Bound

Complete Scheme (General Case)



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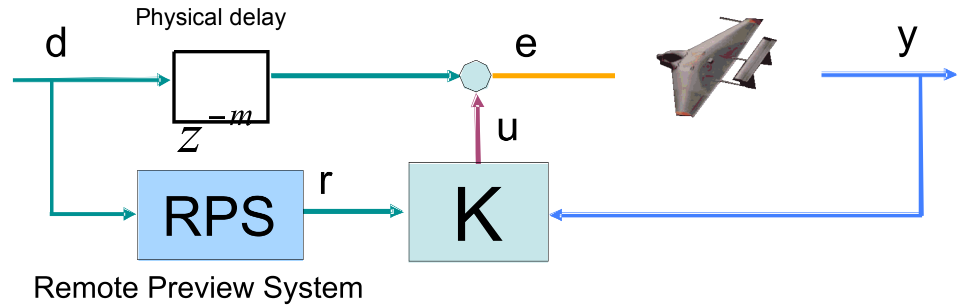
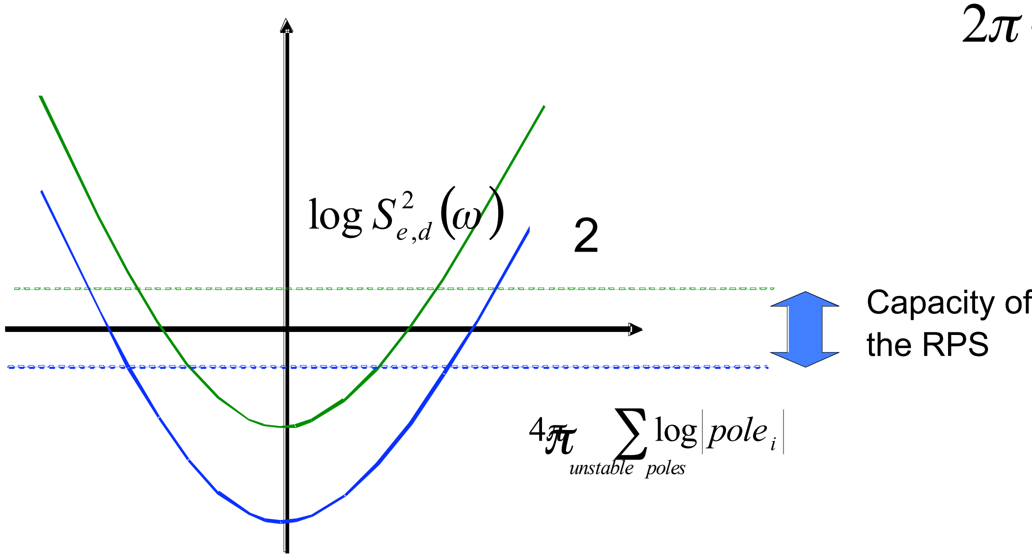
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log(S_{e,d}(\omega)) \geq \qquad \geq \sum_{unstable\ poles} \log|pole_i|$$

$$S_{e,d}^2(\omega) = \frac{F_e(\omega)}{F_d(\omega)} \qquad \boxed{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log(S_{d,e}(\omega))d\omega \geq \sum_{unstable\ poles} \log|pole_i| - C}$$

Motivation for Using Early Warning Information in a Feedback Loop

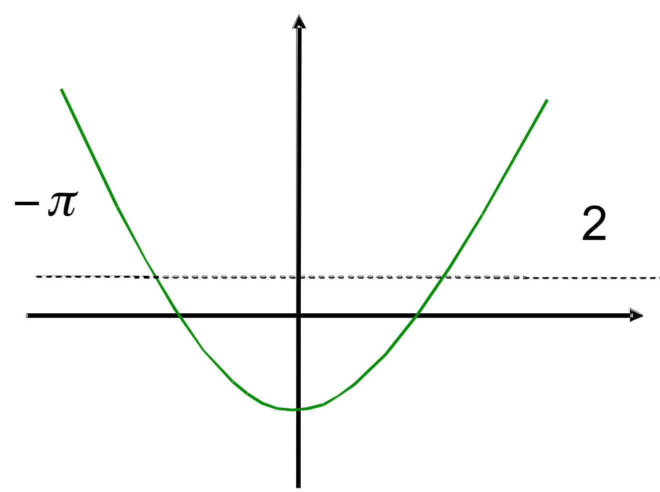


New Bound



Bode

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log S_{e,d}^{2}(\omega)\, d\omega \geq 2 \sum_{unstable\ poles} \log|pole_i|$$



Conclusions

- Using Information Theory, we have derived new bounds in terms of differential entropy, which can be interpreted using standard performance measures.

- Since all quantities are in the units of information, we can characterize fundamental limits arising from capacity constraints.

- New challenging Problems (future directions):

    - How tight are the bounds in general? And what is the role of the delay.

    - These important problems require the interplay between dynamical systems and real-time communication research.

- Potential areas of application:

    - Design of efficient early-warning systems.

    - Networked Control and overhead analysis in IT-MANET.

    - Analysis of hybrid systems. Example: biological systems.