CDS 270-2: Lecture 8-1
Consensus Problem and Algorithms

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- **Goals:**
  - Discuss consensus problems for multi-agent systems
  - Introduce continuous time and discrete time average consensus algorithms
  - Learn the theoretical explanation for consensus behaviors

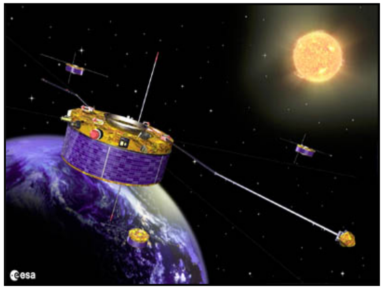
- **Reading:**

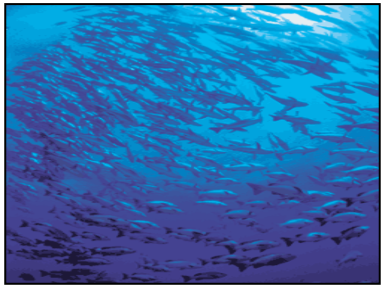
  - “Coordination of groups of mobile autonomous agents using nearest neighbor rules”, A. Jadbabaie, J. Lin, and A. S. Morse, IEEE Transactions on Automatic Control, Vol. 48, No. 6, June 2003, pp. 988-1001

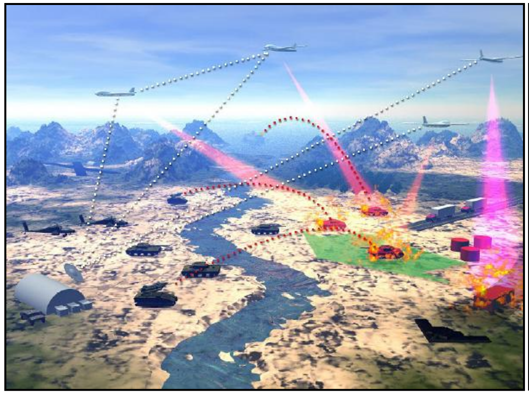
  - “Consensus problems in networks of agents with switching topology and time-delays”, R. Olfati-Saber and R. M. Murray, IEEE Transactions on Automatic Control, vol. 49, 1520-1533, Sept. 2004

Cooperative Control for Multi-Agent Systems

- Applications
  - Formation Control
  - Non-formation cooperation
  - Animal aggregation behavior
- Group level agreement
  - Coordination data or knowledge
  - Pass through a comm. Network
  - Asymptotically approach a common value
- Consensus Problem
  - Collective behavior
  - Distributed algorithms

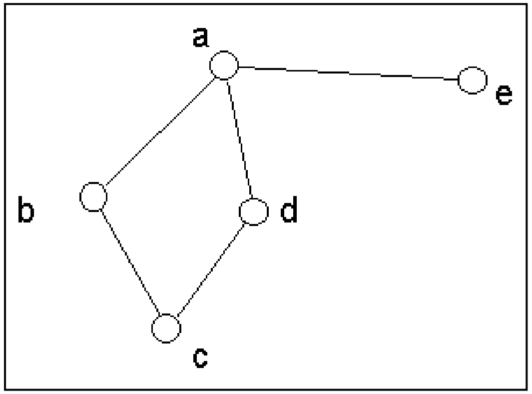






Definition and Background

- States of each agent dynamically evolves under other agents' interactions
- Using a graph to represent the Interaction topology
- Consensus definitions
  - The system reaches a consensus if and only if $x_1 = \cdots = x_n = \eta$
  - The value of consensus state (max., min., average, etc.)
- Consensus Algorithms
  - Self-driving distributed internal rules
  - Difference from synchronization problems



$$\dot{x}_i = f_i(x_i) + u_i(x_i, x_j), x_j \in N_i \Rightarrow \dot{x}_i = u_i(x_i, x_j),\ j \in N_i$$

Definition and Background

- Graph theory
  - Adjacency matrix A

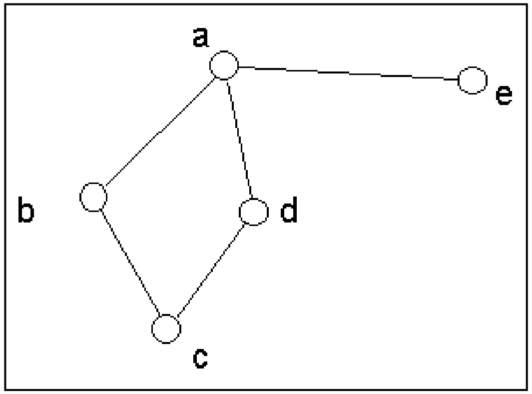
$$a_{ij} = \begin{cases} 1, & (v_i, v_j) \in \mathcal{E} \\ 0, & \text{otherwise.} \end{cases}$$

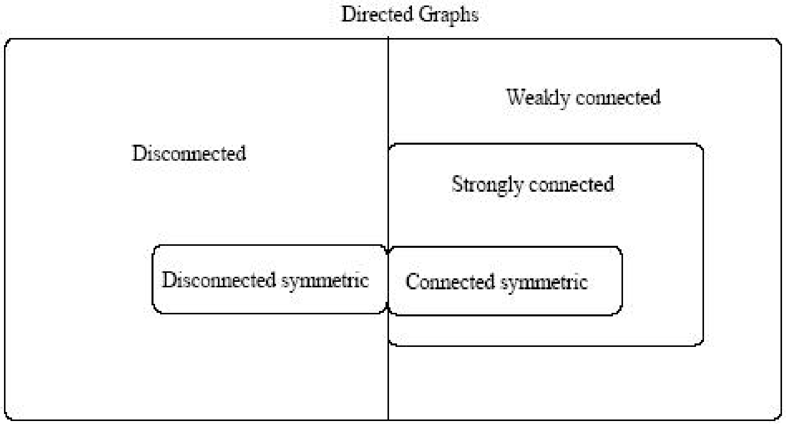
  - Directed or undirected graphs
  - Degree matrix D and Laplacian matrix L

$$L = D - A.$$

  - The set of neighbors N_i
- Average consensus algorithm (nearest neighbor rules)

$$\dot{x}_i = -\sum_{j \in N_i} (x_i - x_j) \quad \text{(A1)}$$





Average consensus algorithm

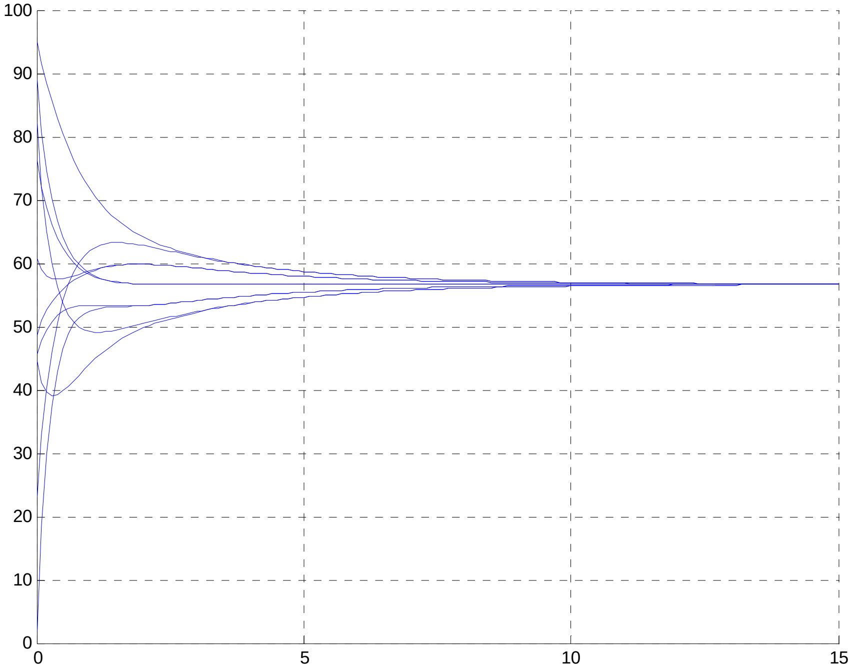
continuous-time case

- Simulation of 10 agents with a connected symmetric graph.

$$\dot{x}_i = -\sum_{j \in N_i} (x_i - x_j) \qquad \text{(A1)}$$

$$\Downarrow$$

$$\lim_{t \to \infty} x_i = \frac{1}{N} \sum_{k=1}^{N} x_k(0)$$



- Theorem:

Consider a multiple agent system with continuous-time average consensus algorithm A1. Assume that the graph G is connected and symmetric. Then A1 globally asymptotically solves the average consensus problem.

Average consensus algorithm

continuous-time case

- Proof:
  - Rewrite the dynamics $\dot{X} = -LX$
  - The spectrum analysis of Laplacian
  - Similarity transformation $Z = V^{-1}X = \begin{bmatrix} 1 & \bar{1}^T \\ \bar{1} & -I_{n-1} \end{bmatrix} X = \begin{bmatrix} \sum x_i \\ x_1 - x_2 \\ \vdots \\ x_1 - x_n \end{bmatrix}$

$$\dot{Z} = -V^{-1}LV \cdot Z = -\begin{bmatrix} 0 & 0 \\ 0 & \widehat{L} \end{bmatrix} \cdot Z$$

  - The consensus state

$$\eta = \sum x_i(0) \Big/ N$$

Average consensus algorithm

discrete-time case

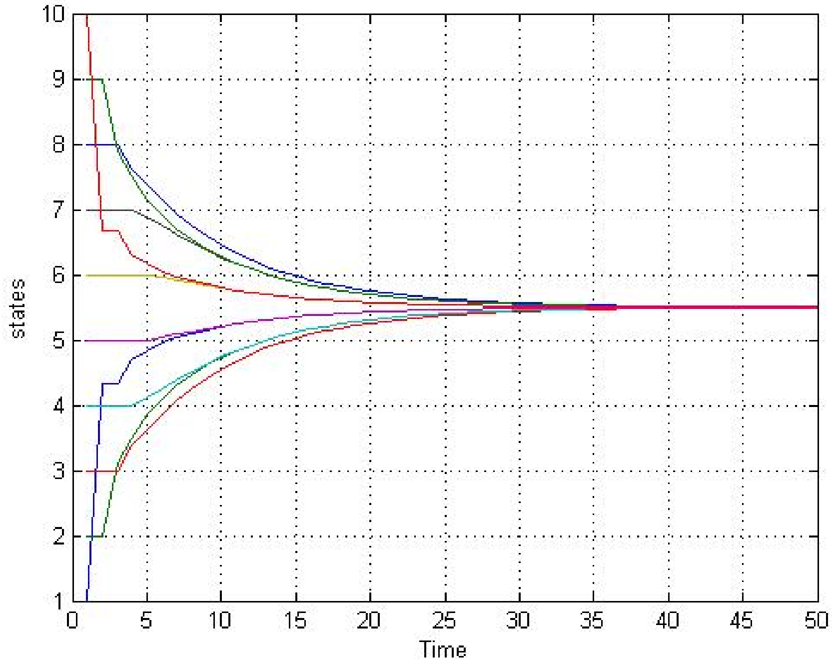
- Simulation of 10 agents with a connected symmetric graph.

$$x_i(k+1) = \frac{1}{|N_i|+1}\left(x_i(k) + \sum_{j \in N_i} x_j(k) \right)$$

(A2)

⇓

$$\lim_{k \to \infty} x_i = \frac{1}{N} \sum_{j=1}^{N} x_j(0)$$



- Theorem:

Consider a multiple agent system with discrete time average consensus algorithm A2. Assume that the graph G is connected and symmetric. Then A2 globally asymptotically solves the average consensus problem.

Average consensus algorithm

distrete-time case

- Proof:
  - Rewrite the dynamics

$$X(k+1) = F \cdot X(k) = (I+D)^{-1} \cdot (A+I) \cdot X(k)$$

  - The spectrum analysis of stochastic matrix

$$\lim_{k \to \infty} F^k = \overline{1} v^T$$

  - The vector v is the left eigenvector of F. If G is connected and symmetric, then

$$v = \overline{1}$$

  - The consensus state

$$\eta = \sum x_i(0) \Big/ N$$

Consensus algorithms on directed graphs

- Directed graphs
  - Adjacency and Laplacian matrices are not symmetric
  - Consensus state

$$\eta = \frac{v^T \cdot X(0)}{v^T \cdot \overline{1}}$$

  - The vector v is the left eigenvector of F
  - Balanced graph (can achieve a consensus)

$$v = \overline{1}$$

  - Strongly connected graph (can achieve a consensus)

$$v = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^T$$

  - Weakly connected graph (cannot guarantee)

Dynamically changing topologies

- The interactive topology may changes over time
- Explore the minimum requirements to reach consensus
- Rough idea: the union of the collection of the interaction graphs across some time intervals are strongly connected frequently enough.
- Theorem

Suppose that the topology G switches according to an infinite time sequence. Then A1 asymptotically solves the consensus problem if there exists an infinite, uniformly bounded time intervals such that the union of the graphs across each interval is strongly connected.

Advanced topic: time delays

- Consider each of the communication link has a time delay

$$h_{ij}(s) = e^{-\tau_{ij} s}$$

- Assume the topology is fixed and delay is uniformly bounded. What's the largest delay margin that A1 can still asymptotically solve the consensus problem.

$$G(s) = (sI + L(s))^{-1} = (sI + e^{-\tau s} L)^{-1}$$

$$\tau^* = \frac{\pi}{2\lambda_{\max}(L)}$$

Advanced topic: convergence speed

- Choosing the optimal weights
  - Assume the graph is connected and symmetric;
  - We has the freedom to choose the weights associated with each edge;
  - Transfer to a semidefinite programming problem
- Changing the interactive topology (for larger scale network)
  - “Small-world” networks have fast convergence speed
  - Using “random rewiring” to change the topology
- Using relay protocols
  - Each agent transfer its neighbors info to its neighbors
  - Add more edges into the original graph
  - Convergence speed V.S. delay sensitivity

Summary

- Introduce the concept of consensus over multi-agent systems. Two popular average consensus algorithms are presented
- Using graph theory and matrix analysis to understand the theoretical explanation of the conditions of average consensus behavior.
- Brief list couple of advanced topics related to consensus problem such as topology switching, communication delays, and convergence speed.