Packet-based Control: the TCP-like case

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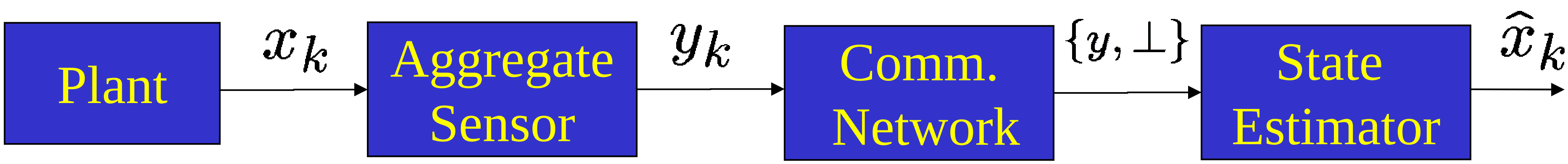
From Monday’s class

Optimal estimation



$$
\begin{aligned}
x_{k+1} &= Ax_k + w_k \\
y_k &= Cx_k + v_k
\end{aligned}
$$

$$p_{\gamma_k}(1) = \bar{\gamma}$$

Modified Kalman Filter Equations

- The arrival of the observation at time t can be modeled as a binary random variable γ_t

$$p_{\gamma_t}(1) = \bar{\gamma}$$

- Kalman Filter Equations

$$\begin{aligned}
\hat{x}_{t+1|t+1} &= \hat{x}_{t+1|t} + \gamma_{t+1} P_{t+1|t} C^T (C P_{t+1|t} C^T + R)^{-1} (y_{t+1} - C \hat{x}_{t+1|t}) \\
P_{t+1|t+1} &= P_{t+1|t} - \gamma_{t+1} \underbrace{P_{t+1|t} C^T (C P_{t+1|t} C^T + R)^{-1}}_{K_{t+1}} C P_{t+1|t}
\end{aligned}$$

- Note:
  - $\hat{x}_{t+1|t+1}$ and $P_{t+1|t+1}$ are random variables, since they depend on γ_{t+1}
  - We need to give a statistical description of $P_{t+1|t+1}$

Approach to the solution

- Let's try to find deterministic difference equations to bound $E[P_{t|t-1}]$

$$\mathbb{E}[P_{t+1|t}] = A\mathbb{E}[P_{t|t-1}]A^T + Q - \lambda \underbrace{\mathbb{E}[AP_{t|t-1}C^T(CP_{t|t-1}C^T + R)^{-1}CP_{t|t-1}A]}_{*}$$

Using Jensen's inequality and monotonicity arguments:

$$A\mathbb{E}[P_{t|t-1}]A^T \geq * \geq A\mathbb{E}[P_{t|t-1}]C^T(C\mathbb{E}[P_{t|t-1}]C^T + R)^{-1}C\mathbb{E}[P_{t|t-1}]A$$

$$S_{t+1} = (1-\lambda)AS_tA' + Q \qquad V_{t+1} = AV_tA' + Q - \lambda AV_tC^T(CV_tC^T + R)^{-1}CV_tA$$

Lower bound

Upper bound

$$S_t \leq E[P_{t|t-1}] \leq V_t$$

Summary

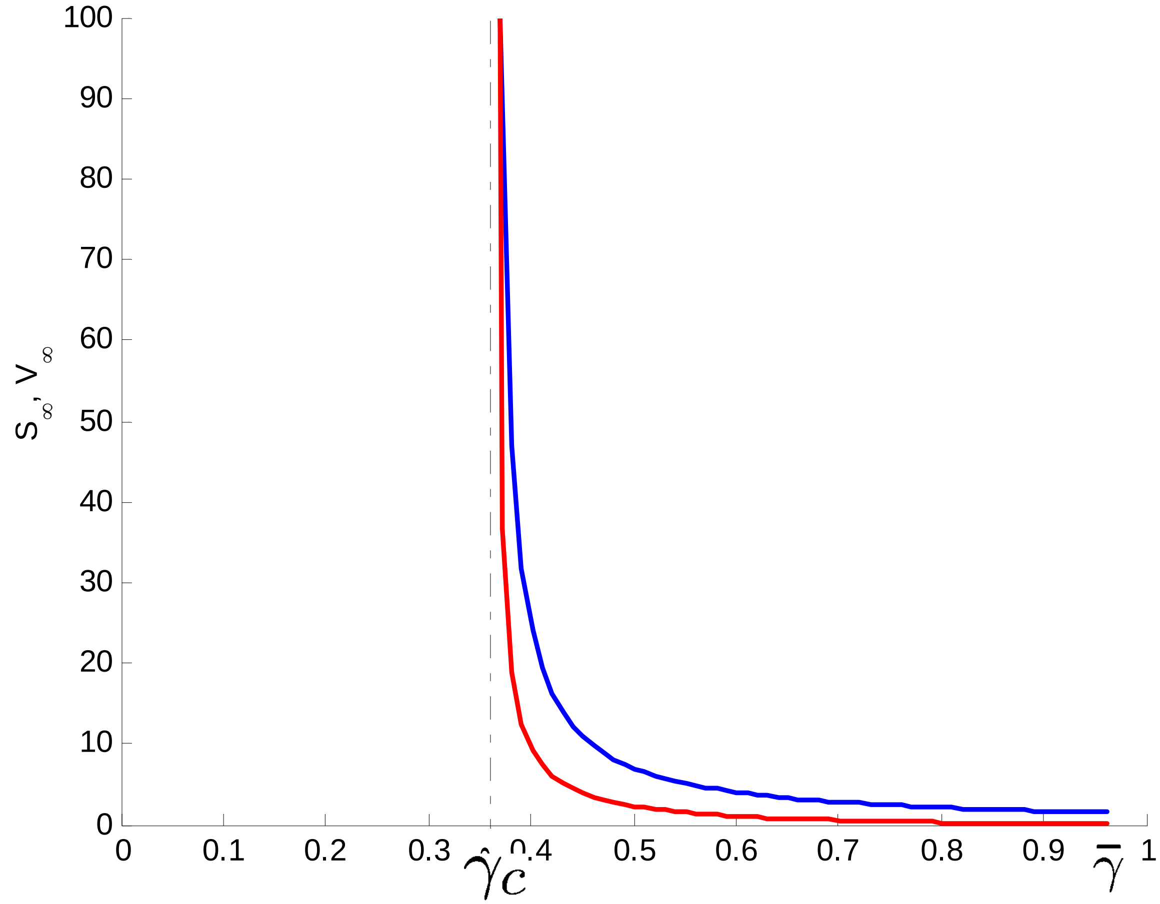
- Optimal dynamic filter among all possible filters
- We can prove the existence of a unique critical value γ_c such that E[P_t] converges for all $\bar{\gamma} > \gamma_c$ and diverges otherwise
- Analytical solution for lower and numerical solution for upper bound for the critical probability

$$\gamma_{min} \leq \gamma_c \leq \gamma_{max}$$

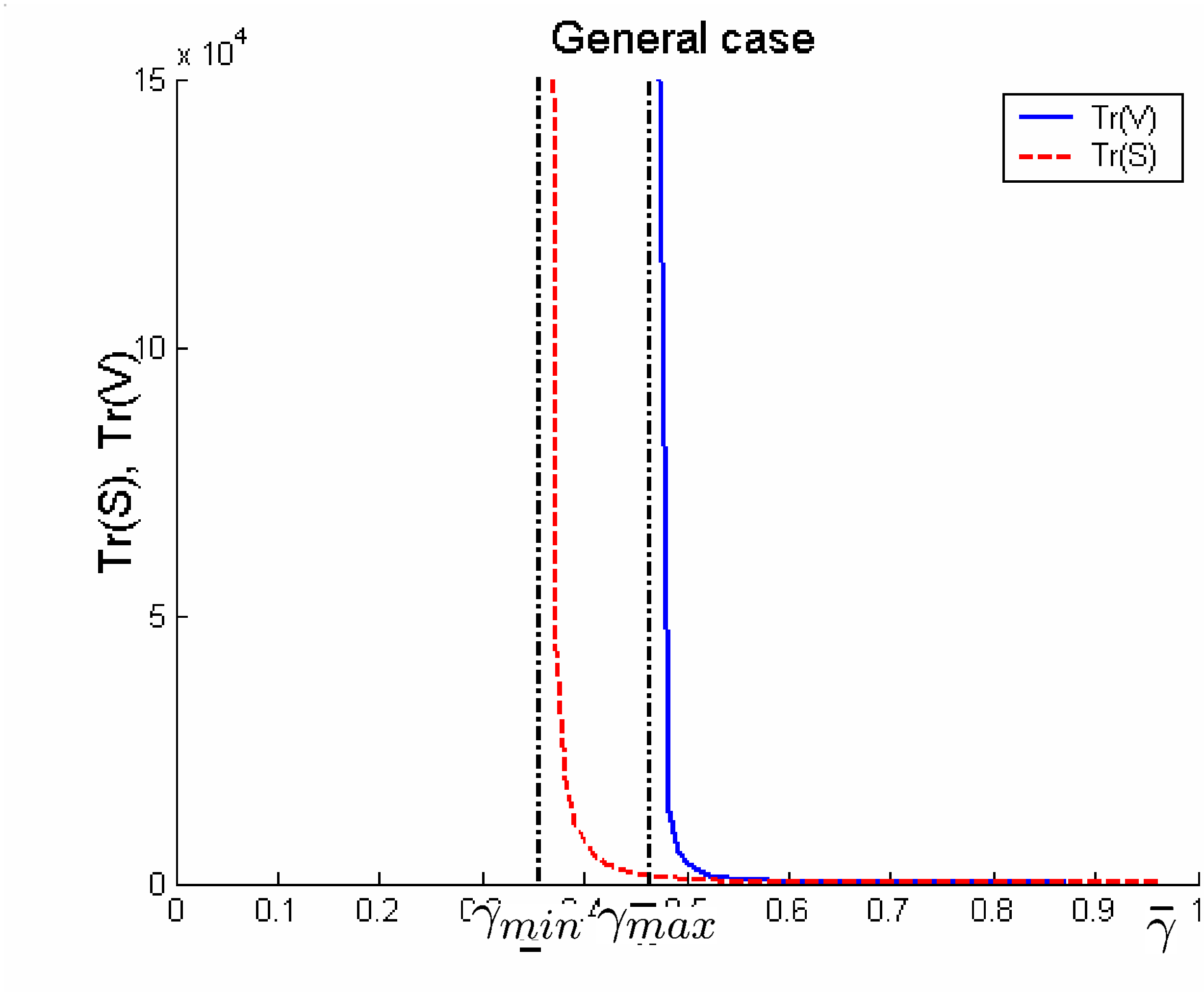
- Numerical solution for lower and upper bounds for the estimation error covariance E[$P_{t|t-1}$]

$$\underline{P} \leq E[P_{t|t-1}] \leq \overline{P}$$

Lower & Upper Bound (Scalar Case)

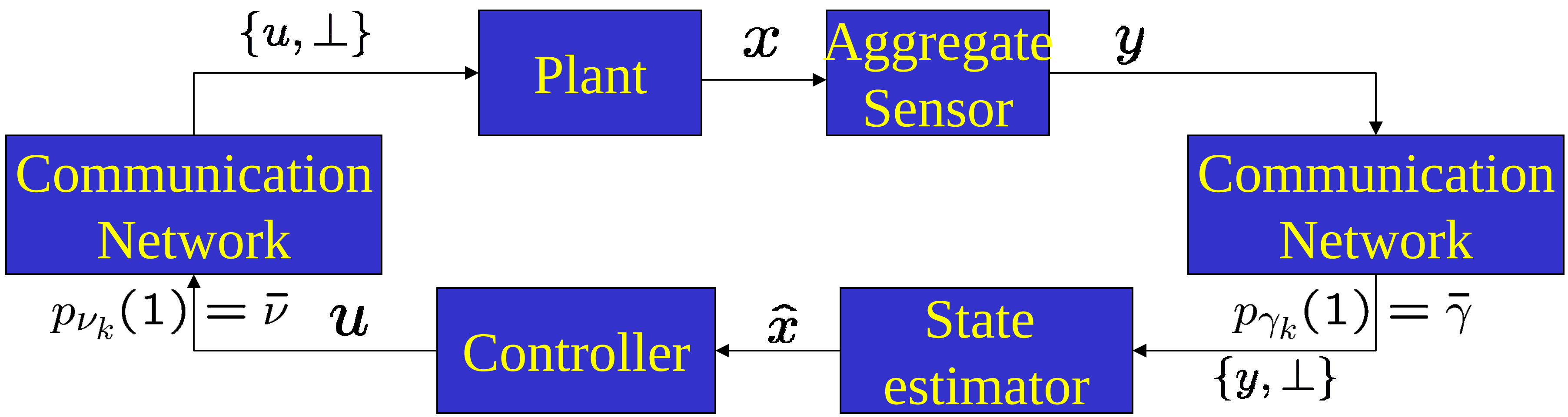


General case



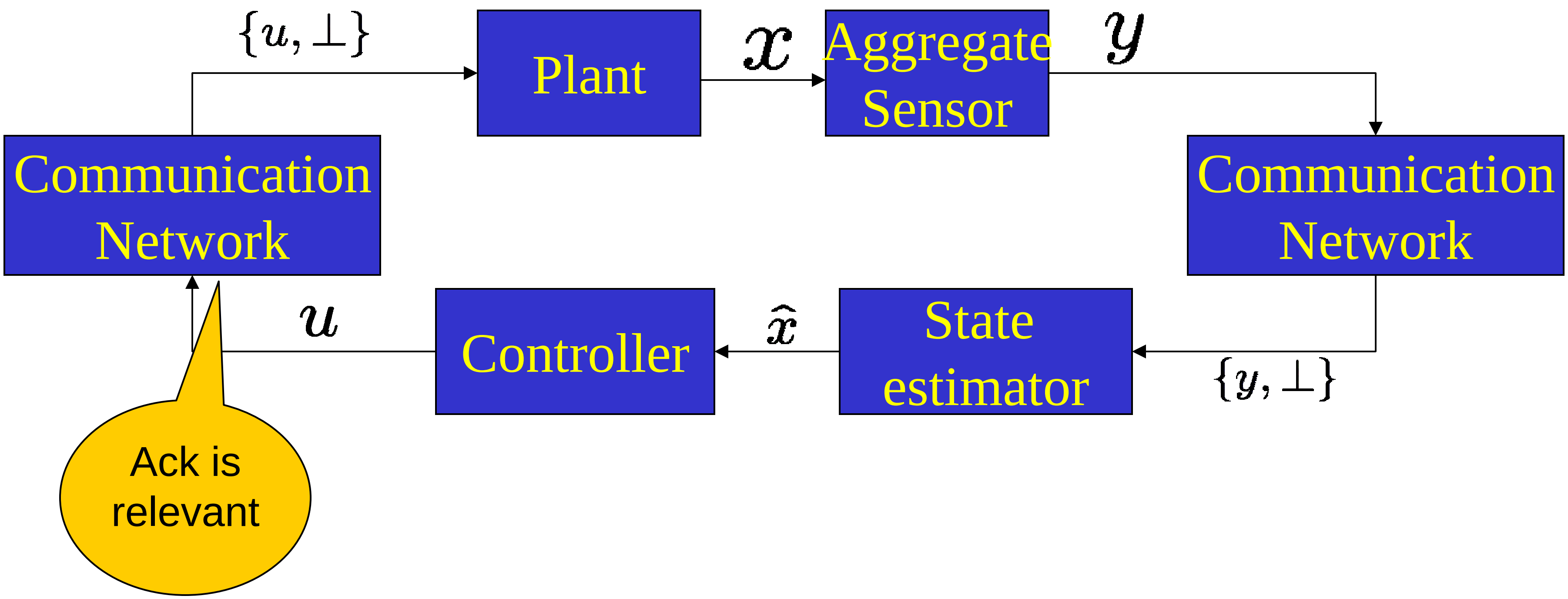
Today

The control problem



- What is the minimum arrival probability that guarantees "acceptable" performance of estimator and controller?
- How is the arrival rate related to the system dynamics?
- Can we design estimator and controller independently?
- Are the optimal estimator and controller still linear?
- Can we provide design guidelines?

LQG control with intermittent observations and control



We'll group all communication protocols in two classes:
TCP-like (acknowledgement is available)
UDP-like (acknowledgement is absent)

Related work

- Imer et al., "*Optimal control of dynamical systems over unreliable communication links.*" in *NOLCOS*, 04
  - C invertible, neither output nor process noise
- Elia, "*Remote stabilization over fading channels*," Systems and Control Letters, 05
  - Model the plant and the controller as deterministic time invariant discrete-time systems connected to zero-mean stochastic structured uncertainty.
  - Time invariant controller, no output or process noise
- Seiler *et al., An H^∞ approach to networked control*," IEEE TAC 2005
  - Bernoulli packet losses only between the plant and the controller, *H^∞ appr.*
- C. N. Hadjicostis et al., "*Feedback control utilizing packet dropping network links,"* CDC 02
  - *Scalar case*
- Other results on the edge of control and information theory
  - Elia, Hespanha, Liberzon, Martins, Mitter, Sahai, Tatikonda,Yung

LQG mathematical modeling

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + w_k \\ u_k^a &= \nu_k u_k^c \\ y_k &= \gamma_k C x_k + v_k, \end{aligned}$$

γ,ν – Bernoulli, indep.

$$J_N(\mathbf{u}^{N-1}, \bar{x}_0, P_0) =$$
$$= \mathbb{E}\Big[x_N' W_N x_N + \textstyle\sum_{k=0}^{N-1} (x_k' W_k x_k + \nu_k u_k' U_k u_k) \Big| \mathbf{u}^{N-1}, \bar{x}_0, P_0\Big]$$

Minimize J_N subject to

$$u_k = g_k(\mathcal{I}_k); \quad \mathcal{I}_k = \begin{cases} \mathcal{F}_k \triangleq \{\mathbf{y}^k, \boldsymbol{\gamma}^k, \boldsymbol{\nu}^{k-1}\}, & \text{TCP-like} \\ \mathcal{G}_k \triangleq \{\mathbf{y}^k, \boldsymbol{\gamma}^k\}, & \text{UDP-like} \end{cases}$$

$$J_N^*(\bar{x}_0, P_0) \triangleq \min_{\mathbf{u_k} = \mathbf{g}_k(\mathcal{I}_k)} J_N(\mathbf{u}^{N-1}, \bar{x}_0, P_0)$$

Estimator Design

Prediction Step

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + \nu_k B u_k$$
$$P_{k+1|k} = A P_{k|k} A' + Q$$

Correction Step

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|t} + \gamma_{k+1} K_{k+1} (y_{k+1} - C\hat{x}_{k+1|t})$$
$$P_{k+1|k+1} = P_{k+1|t} - \gamma_{k+1} P_{k+1|t} C' (C P_{k+1|t} C' + R)^{-1} C P_{k+1|t}$$
$$K_{k+1} = P_{k+1|t} C' (C P_{k+1|t} C' + R)^{-1}$$

LQG Controller Design: TCP-like case

Solution via Dynamic Programming:

1. **Compute the Value Function t=N and move backward**
2. **Find Infinite Horizon by taking N→+∞**

$V_t(x_t)$ – minimum cost-to-go if in state x_t at time t

$$V_N(x_N) \triangleq \mathbb{E}[x_N' W_N x_N \mid \mathcal{F}_N]$$

$$V_k(x_k) \triangleq \min_{u_k} \mathbb{E}[x_k' W_k x_k + \nu_k u_k' U_k u_k + V_{k+1}(x_{k+1}) \mid \mathcal{F}_k]$$

$$J_N^* = V_0(x_0)$$

LQG Controller Design: TCP-like case

We can prove that for TCP the value function can be written as:

$$V_k(x_k) = \mathbb{E}[\ x_k' S_k x_k \mid \mathcal{F}_k] + c_k,\ \ k = N, \ldots, 0$$

with:

$$\begin{aligned} S_k &= A'S_{k+1}A + W_k - \bar{\nu} A'S_{k+1}B(W_k + B'S_{k+1}B)^{-1}B'S_{k+1}A \\ c_k &= \mathbb{E}[c_{k+1} \mid \mathcal{F}_k] + \mathrm{trace}[(\mathrm{A'S_{k+1}A + W_k - S_k)P_{k|k}}] + \mathrm{trace}(\mathrm{S_{k+1}Q}) \end{aligned}$$

$$S_N = W_N, \qquad c_N = 0$$

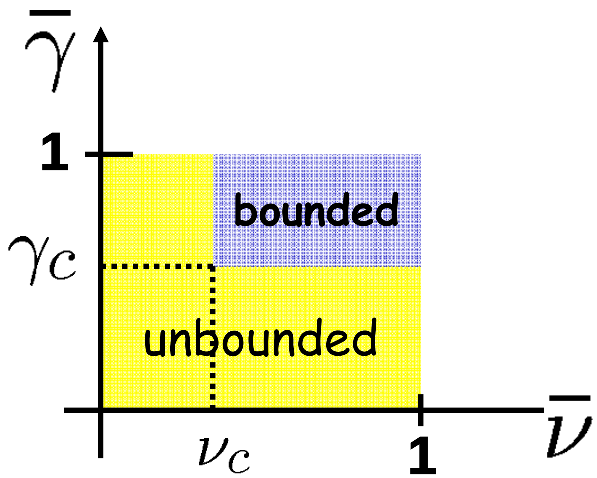
Minimization of v(t) yields:

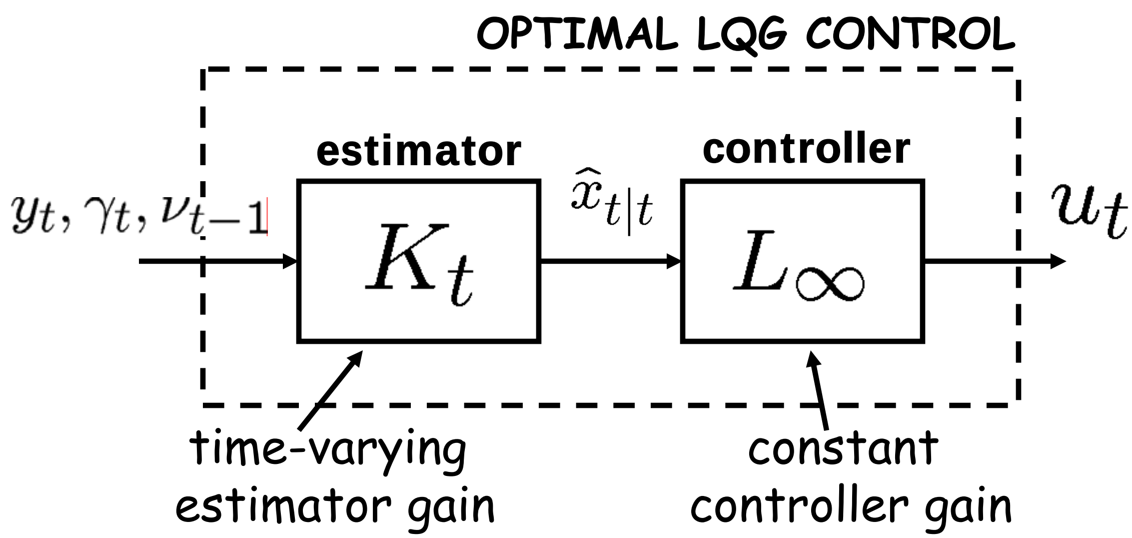
$$u_k = -(B'S_{k+1}B + U_k)^{-1} B'S_{k+1}A\,\hat{x}_{k|k} = L_k\,\hat{x}_{k|k}$$

Infinite Horizon: TCP-like case

LQG averaged cost $\frac{J_N^*}{N}$ is bounded for all N if the following Modified Algebraic Riccati Equations exist:

$$
\begin{aligned}
S_\infty &= A'S_\infty A + W - \bar{\nu}\, A'S_\infty B \overbrace{(B'S_\infty B + U)^{-1} B'S_\infty A}^{L_\infty} \\
P_\infty &= AP_\infty A' + Q - \bar{\gamma}\, AP_\infty C'(CP_\infty C' + R)^{-1} CP_\infty A'
\end{aligned}
$$





Infinite Horizon: TCP case

The critical probability ν_c satisfy the following analytical bounds:

$$p_{min} \leq \nu_c \leq p_{max}$$
$$p_{min} \triangleq 1 - \frac{1}{\max_i |\lambda_i^u(A)|^2}$$
$$p_{max} \triangleq 1 - \frac{1}{\prod_i |\lambda_i^u(A)|^2}$$

where $\lambda_i^u(A)$ are the unstable eigenvalues of A.

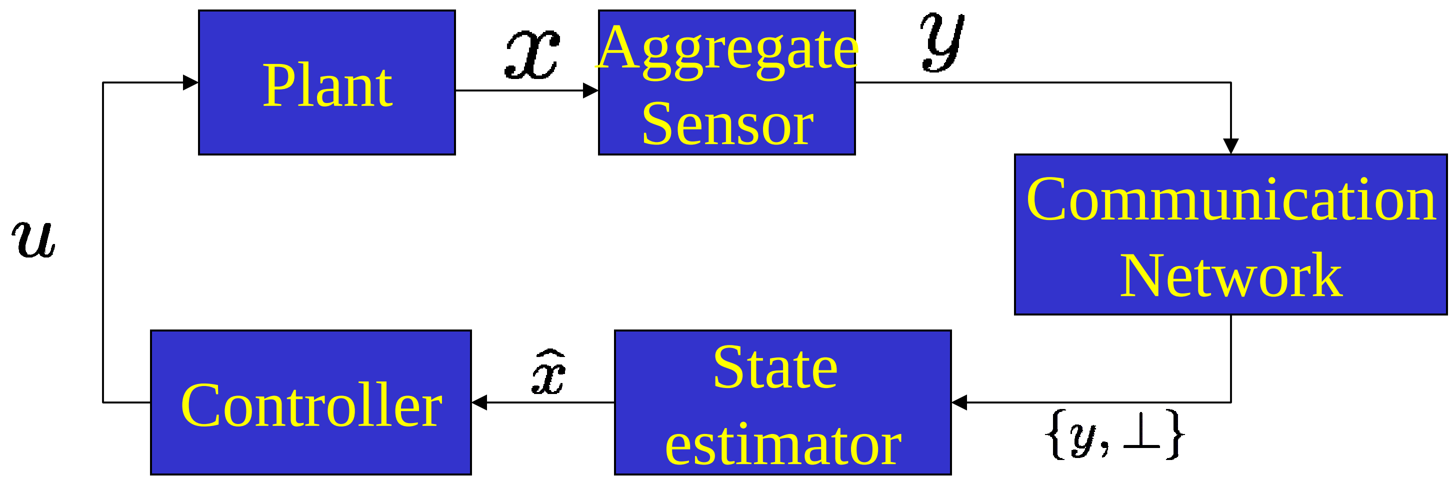
Also, $\nu_c = p_{min}$ when B is invertible

$\nu_c = p_{max}$ when B is rank one

Summary of results

- Finite horizon LQG:
  - The separation principle holds for TCP-like protocols
  - The optimal estimator is the time-varying Kalman Filter
  - The optimal control is linear
  - The optimal cost can be bounded
- Infinite Horizon LQG:
  - There exist critical values γ_c, ν_c below which the closed loop system fails to stabilize the system, and above which we can bound the state
  - We can compute this bound

Special Case: LQG with intermittent observations, $\nu_t = 1, \forall t$



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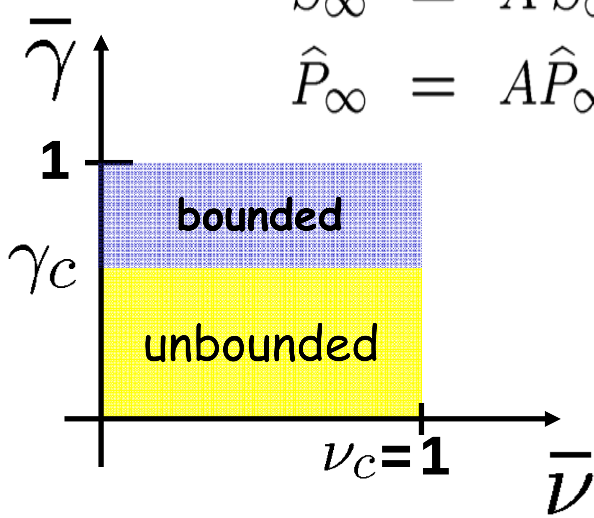
$$S_\infty = A'S_\infty A + W_t - \bar{\nu} A'S_\infty B(W_t + B'S_\infty B)^{-1} B S_\infty A$$

$$\hat{P}_\infty = A\hat{P}_\infty A' + Q - \bar{\gamma} A\hat{P}_\infty C'(C\hat{P}_\infty C' + R)^{-1} C\hat{P}_\infty A'$$



Ideas

- Look at different control strategies:
  - Hold previous value of control
  - Send several controls per time step
  - Look at mixed TCP, UDP strategies
  - Look at different cost functions