CDS 140b: Control of bifurcations & limit cycles

RMM 19 Feb 08

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$\boxed{\text{Background: }{}^{\text{NL}}\text{ Stabilization}}$

Given a nonlinear control system

$$\dot{x} = f(x) + g(x)u \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\quad f, g \text{ smooth}$$

when can we find a smooth feedback $u = \alpha(x)$ that makes the closed loop system asymptotically stable:

$$(*) \qquad \dot{x} = f(x) \cancel{\alpha(x)} + g(x)\alpha(x)$$

For ~~ev~~ any $\varepsilon > 0$, $\exists\ \delta > 0$ s.t. $\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon\ \forall t$ — Stable i.s.L.

and $\lim_{t\to\infty} x(t) = 0$

Defn A smooth, proper, positive definite function $V: \mathbb{R}^n \to \mathbb{R}$ is called a <u>control Lyapunov function</u> (CLF) if

$$\inf_{u \in \mathbb{R}^m} \ L_f V(x) + L_{g_1} V(x) u_1 + \cdots + L_{g_m} V(x) < 0$$

for all $x \neq 0$. [Proper: $V(x) \to \infty$ as $\|x\| \to \infty$; $L_f V = \frac{\partial V}{\partial x} f(x)$]

<u>Thm</u> (Artstein, 1983) If there exists a CLF $V(x)$ for system $(*)$ then $\exists\ u = \alpha(x)$ with α smooth on $\mathbb{R}^n \setminus \{0\}$ that globally asymptotically stabilizes $(*)$

<u>Remarks</u>

1. We will prove via explicit construction (Sontag, 1989)
2. For a linear system $\dot{x} = Ax + Bu$, this is equivalent to stabilizability (exercise). Can extend to linearized.
3. Conditions can be shown to be necessary as well

<u>Defn</u> A CLF satisfies the <u>small control property</u> if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $x \neq 0$ and $\|x\| < \delta$, there exists u with $\|u\| < \varepsilon$ such that

$$L_f V(x) + \sum_i L_{g_i} V(x)\, u_i < 0$$

<u>Thm</u> Suppose V is a CLF for $(*)$ with V satisfying the small control property. Then $\exists$ a ~~smooth~~ feedback $u = \alpha(x)$ ~~the~~ that is smooth on $\mathbb{R}^n \setminus \{0\}$ and continuous everywhere that stabilizes the closed loop.

<u>Pf</u> Consider first the scalar case

$$\dot{x} = f(x) + g(x)u \qquad a(x) := L_f V$$
$$b(x) := L_g V$$

Since V is a CLF, $b(x) = 0 \Rightarrow a(x) < 0$. Let $A = a(x)$ and $B = b(x)$ define a family of linear systems $\dot{z} = Ax + Bu$. V CLF $\Rightarrow$ (A, B) is stabilizable for each x. (Remember this is scalar for now).

Define $\alpha(x) = -\dfrac{a(x) + \sqrt{a^2 + b^4}}{b}$. Claim $\alpha(x)$ is stabilizing, ~~smooth~~ continuous everywhere and smooth away from 0.

$$\dot{V} = \cancel{\frac{\partial V}{\partial x}\left(a(x) + \left(-a(x) + \sqrt{a^2 + b^4}\right)\right) = \frac{\partial V}{\partial x}\sqrt{a^2 + b^4}}$$
$$= L_f V + L_g V\, \alpha(x) = -\sqrt{a^2 + b^4} < 0$$

The control law $\alpha(x)$ is clearly smooth when $b(x) \neq 0$

To show continuity when $b(x) = 0$, use ε-δ proof (see Sontag).

~~Rayleigh~~

Vector version

$$a(x) = L_f V \qquad B(x) = (b_1(x), \ldots, b_m(x))$$

$$b_i(x) = L_{g_i} V \qquad \beta(x) = \| B(x) \|^2 = \sum b_i^2(x)$$

$$u_i = \alpha_i(x) = -b_i(x) \frac{a(x) + \sqrt{a^2 + \beta^2}}{\beta}$$

Can show this gives $\dot{V} = -\sqrt{a^2(x) + \beta^2(x)} < 0$ and get continuity by ε-δ argument $\square$

Remarks

1. Why b^4 inside sqrt? A: Suppose we choose

$$\alpha(x) = -\frac{a + \sqrt{a^2 + b^2}}{b} \approx \frac{a + \sqrt{|a^2|} + |b|}{b} = \frac{|b|}{b} \text{ when } a < 0$$

By using b^4, we get $\alpha(x) \to 0$ as $x \to 0$

2. The main reason this formula is useful is ~~to~~ in proofs. No gaurantees of robustness or performance

3. ~~Key takeaway~~ No assumption of linearity or even that the linearization is ~~control~~ reachable $\Rightarrow$ very general result.

4. Bad news: finding CLFs is not a convex problem $\Rightarrow$ hard in general.

Strongly nonlinear systems $\quad \dot{x} = f(x,u)$

Thm (Brockett) ~~If~~ Let $B_{\varepsilon_1,\varepsilon_2}$ be a ball of size $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ in $\mathbb{R}^{n \times m}$. If $f(B_{\varepsilon_1}, B_{\varepsilon_2})$ does not contain a nbd of the origin, then there does not exist a continuous stabilizer $u = \alpha(x)$.

Example

$$\dot{x}_1 = u_1 \qquad \dot{x}_2 = u_2 \qquad \dot{x}_3 = x_2 u_1 - x_1 u_2 \qquad f(x,u) = \begin{bmatrix} u_1 \\ u_2 \\ x_2 u_1 - x_1 u_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ \varepsilon \end{bmatrix}$$

$\Rightarrow$ No continuous stabilizer (and no CLF)

Remarks

1. More generally, $\dot{x} = g(x) u$ cannot be C^0 stabilized (even if it is reachable, in a nonlinear setting). Note that these systems also have a nonreachable linearization.
2. If we allow $u = \alpha(x,t)$, can get C^∞ asymptotic stabilizers, but not C^1 exponential stabilizers (Coron)
3. If we allow non-Lipschitz feedback, we can get exponential stability in individual states (M'Closkey)

$$u_1 = -x_1 + \sqrt{|x_3|}\,\sin \omega t$$
$$u_2 = -x_2 + \sqrt[5]{x_3}\,\cos \omega t$$