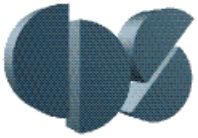
Lecture 5: Distributed Control Systems



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17 March 2009

Goals:

- Define centralized versus decentralized versus distributed control
- Describe analysis results in stability of distributed control systems, with a focus on the role of topology of the information flow
- Summarize synthesis techniques for distributed control systems

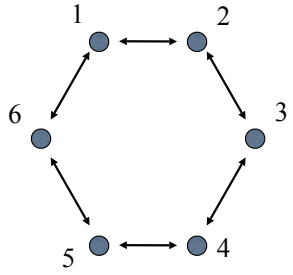
Reading:

- J. A. Fax and R. M. Murray, "Information flow and cooperative control of vehicle formations", *IEEE T. Automatic Control*, 2004.
- V. Gupta, B. Hassibi and R. M. Murray, “A sub-optimal algorithm to synthesize control laws for a network of dynamic agents”, *Int’l J. Control*, 2005
- R. D'Andrea and G. E. Dullerud,"Distributed Control Design for Spatially Interconnected Systems", *IEEE T. on Automatic Control,* 2003.
- M. Rotkowitz and S. Lall,, "A Characterization of Convex Problems in Decentralized Control", *IEEE T. on Automatic Control,* 2006.

Sample Problem: Formation Stabilization

Goal: maintain position relative to neighbors

- “Neighbors” defined by graph
- Assume only sensed data for now
- Assume identical vehicle dynamics, identical controllers?



Example: hexagon formation

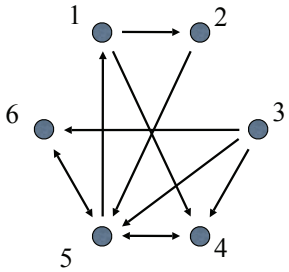
- Maintain fixed relative spacing between left and right neighbors

$$e_i = \sum_{j \in N_i} w_j (y_i - y_j - h_{ij})$$

relative position

weighting factor

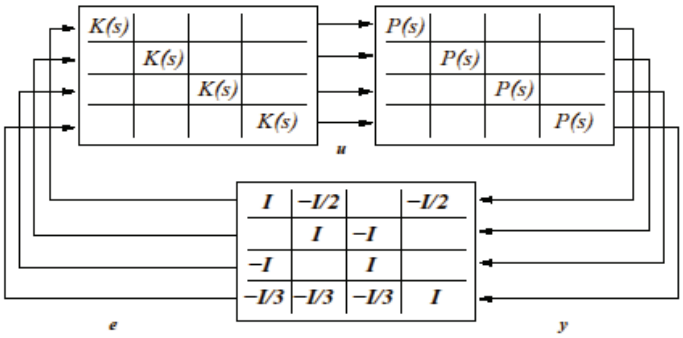
offset

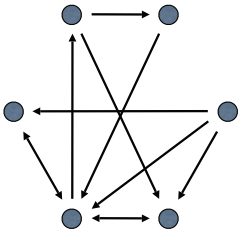


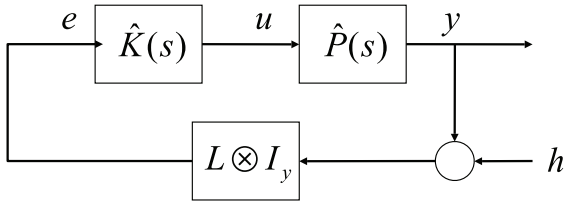
Can extend to more sophisticated “formations”

- Include more complex spatio-temporal constraints

Mathematical Framework







Analyze stability of closed loop

- Interconnection matrix, *L*, is the *weighted Laplacian* of the graph
- Stability of closed loop related to eigenstructure of the Laplacian

Fax and M
IEEE TAC 2004

Stability Condition

Agent dynamics

$$\dot{x}^i = Ax^i + Bu^i$$
$$y^i = Cx^i$$

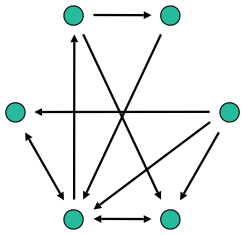
Control law

$$\dot{\xi}^i = F\xi^i + Gz^i$$
$$u^i = H\xi^i + Kz^i$$

Weighted error

$$z^i = \frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} (y^i - y^j)$$

- Agents have identical, linear dynamics
- Control law is dynamic compensator based on sum of relative errors on neighbors
- Can also allow feedback on internal state (fold into A)



$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & -1 & 0 \\ -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Theorem *Let L be the weighted Laplacian of the communications graph $\mathcal{G}$. The closed loop system is (neutrally) stable iff the systems*

$$\dot{x}^i = Ax^i + Bu^i \qquad \dot{\xi}^i = F\xi^i + Gz^i$$
$$z^i = \lambda_j Cx^i \qquad u^i = H\xi^i + Kz^i$$

are stable for each eigenvalue λ_i of L.

Remarks

- Stability is based on check of *n* decoupled systems
- λ_i plays the role of a "loop gain": describes how your output affects your input

Sketch of Stability Proof

$$\dot{x}^i = Ax^i + Bu^i$$
$$z = L\widehat{C}x$$
$$\dot{\xi}^i = F\xi^i + Gz^i$$
$$u^i = H\xi^i + Kz^i$$

Notation

- $\widehat{A} = I_N \otimes A$: block diagonal matrix with A as elements
- $A_{(n)} = A \otimes I_n$: replace elemnts of A with $a_{ij} I_n$
- For $X \in \mathbb{R}^{r\times s}$ and $Y \in \mathbb{R}^{N\times N}$, $\widehat{X}Y_{(s)} = \widehat{Y}X_{(r)}$

Let T be a Schur transformation for L, so that $U = T^{-1}LT$ is upper triangular. Transform the (stacked) process states as $\tilde{x} = T_{(n)}x$ and the (stacked) controller states as $\tilde{\xi} = T_{(n)}\xi$. The resulting dynamics become

$$\frac{d}{dt}\begin{bmatrix}\tilde{x}\\ \tilde{\xi}\end{bmatrix} = \begin{bmatrix}\widehat{A} + \widehat{B}\widehat{K}\widehat{C}U_{(n)} & \widehat{B}\widehat{H}\\ \widehat{G}\widehat{C}U_{(n)} & F\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \tilde{\xi}\end{bmatrix}.$$

Slightly simpler version: just prove for output feedback case with no xi variables. This gets rid of the non-triangular block structure.

block

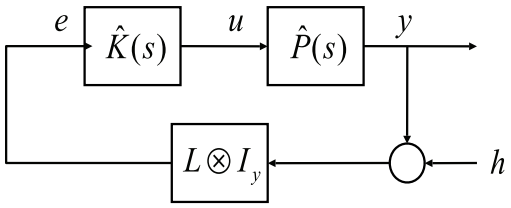
This system is upper triangular, and so stability is determined by the elements on the (block) diagonal:

$$\frac{d}{dt}\begin{bmatrix}\tilde{x}_j\\ \tilde{\xi}_j\end{bmatrix} = \begin{bmatrix}A + BKC\lambda_j & BH\\ GC\lambda_j & F\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \tilde{\xi}\end{bmatrix}.$$

Alternatively, rearrange states as (x, xi) for each block and then you can recover the block structure.

This is equivalent to coupling the process and controller with a gain λ_i.

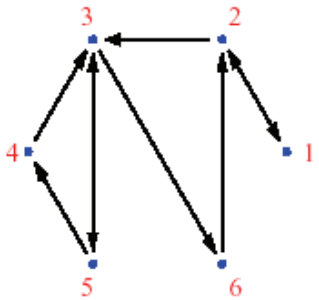
Frequency Domain Interpretation

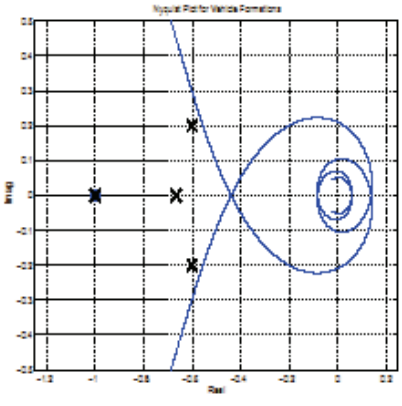


Theorem *The closed loop system is (neutrally) stable iff the Nyquist plot of the open loop system does not encircle* $-1/\lambda_i(L)$, *where* $\lambda_i(L)$ *are the nonzero eigenvalues of L.*

Example

$$P(s) = \frac{e^{-s\tau}}{s^2} \qquad K(s) = K_d s + K_p$$





Introduction to Distributed Control

Vijay Gupta

March 16, 2009

Abstract

In this lecture, we take a look at the problem of distributed control. We will begin by seeing why the problem is hard. Then we will look at one obvious approach towards solving the problem. Other approaches to the problem will also be mentioned.

Contents

1 Introduction

Distributed Control is a very widely used and ill-defined term. We will consider one possible way of defining such systems.

Conventional controller design problem assumes that all the controllers present in the system have access to the same information. Thus, typically, the controller design problem if to design a controller K for a plant P such that some performance specification $\min \parallel f(P,K) \parallel$ is met[1]. As an example, the classical LQG problem can be stated as follows. Given a plant P of

[1]There is usually also an additional specification that K should stabilize the plant.

the form

$$x(k+1) = Ax(k) + Bu(k) + w(k),$$

design a controller that generates control inputs $u(k)$ as a causal function of the measurements

$$y(k) = Cx(k) + v(k)$$

and minimizes a quadratic cost function of the form

$$J = E \sum_{k=0}^{K} \left[x(k)^T Q x(k) + u(k)^T R u(k) \right].$$

The noises $w(k)$ and $v(k)$ are assumed white and Gaussian.

In the corresponding distributed control problem, multiple plants of the form

$$x_i(k+1) = A_i x_i(k) + \sum_{i \neq j} A_{ij} x_j(k) + B_i u_i(k) + w_i(k)$$

are present. If the terms A_{ij} are all zero, the plants are said to be *dynamically uncoupled*. Each plant i has access to (possibly noisy) observations about the states of a set of other agents. We refer to this set as the out-neighbors of the agent i and denote it as $\mathcal{N}_i$[2]. For simplicity, throughout this lecture we will assume that each agent can access the states of all its out-neighbors perfectly. Denote by $x(k)$ the vector formed by stacking the states of all the individual agents $x_i(k)$'s and define vectors $u(k)$ and $w(k)$ similarly. The aim is to design the control laws of the individual agents to minimize (say) a quadratic cost function of the form

$$J = E \sum_{k=0}^{K} \left[x(k)^T Q x(k) + u(k)^T R u(k) \right],$$

where in general Q and R are full. The additional constraint is that each control input $u_i(k)$ can only depend on the states of agents in the set $\mathcal{N}_i$. If we try to minimize the cost function directly, we will come up with a control law of the form $u(k) = F(k)x(k)$ where the matrix $F(k)$ is full in general and thus does not satisfy this *topology constraint*. Solving the problem in the presence of this constraint is a much harder problem.

[2]By convention we assume that $i \in \mathcal{N}_i$.

Thus, in general, a distributed control problem can be stated in the form [22]

$$\begin{aligned} &\text{minimize } \| f(P,K) \| \\ &\text{subject to } K \text{ stabilizes } P \\ &\qquad K \in S, \end{aligned} \tag{1}$$

where S is a subspace[3]. For a general linear time-invariant plant P and subspace S, there is no known tractable algorithm for computing the optimal K. In the next section, we try to see why this problem is hard. We will restrict ourselves to the case of linear plants and quadratic costs from now on.

2 Information Pattern

While the problem stated in 1 is tractable (at least for the special LQ case we are concentrating on) if the subspace constraint is not present, imposing the constraint that K lie only in the subspace S renders the problem open in general. One of the earliest works that pointed out that just the assumptions of a linear plant, quadratic cost and Gaussian noises are not sufficient to obtain the solution was the famous counter-example provided by Witsenhausen [1] (see also [3]). The problem was originally posed in terms of two stages. We can view them as two agents in our setting. Consider $x(0)$ and v to be two independent scalar random variables. At the first stage, the random variable $x(0)$ is viewed. Thus the output equation is

$$y(0) = x(0).$$

Based on this observation, a control input $u(0)$ is calculated and applied. The state then evolves to

$$x(1) = x(0) + u(0).$$

At the next stage, the output equation is

$$y(1) = x(1) + v.$$

A control input $u(1)$ that depends on $y(1)$ is then calculated and applied to obtain

$$x(2) = x(1) - u(1).$$

[3]The way we have defined the problem makes it very similar to the problem of finding a *structured controller* for a plant.

The objective is to minimize the cost function given by

$$J = k^2 u(0)^2 + x(2)^2.$$

The admissible controllers are

$$\begin{aligned} u(0) &= \gamma_0 \left(y(0) \right) \\ u(1) &= \gamma_1 \left(y(1) \right), \end{aligned}$$

where γ_0 and γ_1 are Borel functions.

Note that if $u(1)$ were allowed the knowledge of $u(0)$, the problem can be solved using LQG like methods. However, in the present case, there is information to signal and the observation $y(1)$ can be used to signal that information. There is a trade-off between maximizing the information available (signaling) and minimizing the use of control at the first stage. Note the form of the cost function. At the second stage, all we are penalizing is $x(2)$ which is calculated as

$$x(2) = x(1) - u(1).$$

The controller needs to estimate $x(1)$ from $y(1)$ as best as it can, so that it can set $u(1)$ close to $x(1)$. On the other hand, at the first stage, we do not penalize the state $x(1)$ and hence the controller can choose $u(0)$ arbitrarily without worrying about $x(1)$. Thus we are asking for $x(1)$ to be

1. low entropy, so that it can be easily predicted.

2. high energy, so that the noise v does not affect it much.

Affine controllers would mean Gaussian random variables and for Gaussian variables these two aims are in direct opposition. Non-linear controllers thus can achieve better performance.

While it is known that a nonlinear controller can achieve much better performance than any linear controller, the optimal controller of this problem is still unknown. As an instance, for $k = 0.1$, the best possible affine control law gives a cost of 0.99, while non-linear controllers are possible which drive the cost as close to zero as desired. It can also be shown that the cost function is no longer convex in the controller variables, hence the problem is hard to solve numerically.

This simple counterexample is important since it shows that even for linear plants, quadratic costs and Gaussian noises, linear controls may not be optimal and the problem may be very difficult to solve. The additional piece

that makes the conventional control problem simple is that of the *information pattern*. Informally, the information pattern is a representation of the information set that each decision maker in the problem (e.g. the controller) has access to at every time step when it makes the decision (e.g. calculates the control input). As an example, in the conventional LQG control problem, the controller at time step k has access to all the measurements $y(0)$, $y(1)$, $\cdots$, $y(k-1)$ as well as all the previous control inputs $u(0)$, $u(1)$, $\cdots$, $u(k-1)$. This is called a *classical information pattern*[4]. As Witsenhausen's counterexample shows, a non-classical information pattern can render a control problem intractable. Since in a distributed control problem, different controllers have access to different information sets, the information pattern is not classical and hence the problem is inherently difficult. It can be shown [4, 5], e.g., that the problem of finding a stabilizing decentralized static output feedback is NP-complete.

Since the general problem is difficult, there are two main approaches that have been proposed:

1. Identifying sub-optimal solutions.
2. Identifying special conditions or information patterns under which the problem can be solved.

We now look at these approaches in a bit more detail.

3 Sub-optimal Controller Synthesis

In this section, we will take a look at some of the approaches that have been suggested to implement sub-optimal controllers for arbitrary interconnection topology (and hence arbitrary sub-space constraints) on the controller.

Perhaps the approach that is most easy to understand is the one inspired by the design of reduced-order controllers (e.g., [6]). This approach was used to obtain numerical algorithms for solving the optimal linear control with arbitrary number of free parameters for the infinite horizon case in, e.g., [7, 8]. We will consider the version presented in [9].

Consider N dynamically uncoupled agents evolving as

$$x_i(k+1) = A_i x_i(k) + B_i u_i(k),$$

where the control of the i-th agent can depend linearly on its own state value and the states of a specified set of other agents. On stacking the states of

[4]Alternatively, the information pattern has *total recall*.

all the agents, the system evolves as

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ u(k) &= Fx(k), \end{aligned}$$

where F is a matrix that incorporates the interconnection information. In particular, F has a block structure, with the (i,j)-th block zero if agent i cannot obtain the information about agent j to calculate its state value. Thus F is constrained to lie in a particular space. Assume that the initial condition $x(0)$ is random and Gaussian with mean zero and covariance $R(0)$. We wish to find the constrained control law F that minimizes the cost function

$$J = E\left[\sum_{k=0}^{\infty}\{x^T(k)Qx(k) + u^T(k)Ru(k)\}\right].$$

Assume that a F exists in the required space, such that $A+BF$ is stable. Then, for that F, the cost function is given by

$$J = E\left[x^T(0)Px(0)\right],$$

where P satisfies the discrete algebraic Lyapunov equation

$$P = (Q + F^T RF) + (A + BF)^T P(A + BF).$$

Thus the cost is given by $J = \text{trace}(PR(0))$ with $R(0)$ as the initial covariance.

The case when noise is present can also be expressed similarly. Suppose that the system evolves as

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + w(k) \\ u(k) &= Fx(k), \end{aligned}$$

where F is chosen to minimize the cost function

$$J = \lim_{k\to\infty} E\left[x^T(k)Qx(k) + u^T(k)Ru(k)\right].$$

As an exercise, prove that the cost can now be written as $J = \text{trace}(PR_w)$ where R_w is the covariance of noise $w(k)$. Note that because F is stable, the initial condition $R(0)$ would not affect the cost function.

3.1 Stabilizability

Two questions arise immediately:

1. Is it possible to stabilize the system using information from other agents when the agents are individually not stable. In other words, if an agent is unstable, can the system be stabilized by the exchange of information between different agents?

2. Are some topologies inherently unstable in that even if the agents are stable, the information flow will always make it impossible to stabilize the formation?

The following result [9, 10]) answers these questions.

Proposition 1. *Consider a system of interconnected dynamically uncoupled agents as defined above.*

1. *The system is controllable if and only if each individual agent is controllable.*

2. *The system is stabilizable if and only if each individual agent is stabilizable.*

Proof. We present the proof for the case of identical agents. The case of non-identical agents is similar and is left as an exercise. Suppose there are N agents each with state-space dimension m with state matrices Φ and Γ. Thus the entire system has state-space dimension Nm and system matrices

$$\begin{aligned} A &= I \otimes \Phi \\ B &= I \otimes \Gamma, \end{aligned}$$

where I is the identity matrix of suitable dimensions and $\otimes$ represents the Kronecker product. For controllability of the system, we thus want the following matrix to have rank Nm

$$M_1 = \begin{bmatrix} I \otimes \Gamma & (I \otimes \Phi)(I \otimes \Gamma) & (I \otimes \Phi)^2(I \otimes \Gamma) & \cdots & (I \otimes \Phi)^{Nm-1}(I \otimes \Gamma) \end{bmatrix}.$$

Using the standard property of Kronecker product

$$(a \otimes b)(c \otimes d) = ac \otimes bd,$$

we can rewrite M_1 as

$$M_1 = \begin{bmatrix} I \otimes \Gamma & (I \otimes \Phi\Gamma) & (I \otimes \Phi^2\Gamma) & \cdots & (I \otimes \Phi^{Nm-1}\Gamma) \end{bmatrix}.$$

This matrix has rank Nm if and only if the following matrix has rank m

$$M_2 = \begin{bmatrix} \Gamma & \Phi\Gamma & \Phi^2\Gamma & \cdots & \Phi^{Nm-1}\Gamma \end{bmatrix}.$$

Since Φ is an $m \times m$ matrix, the equivalent condition is that the matrix

$$M_3 = \begin{bmatrix} \Gamma & \Phi\Gamma & \Phi^2\Gamma & \cdots & \Phi^{m-1}\Gamma \end{bmatrix}$$

has rank m. But M_3 being rank m is simply the condition for the individual agent being controllable. Thus the system is controllable if and only if each individual agent is controllable. This proves the first part. The proof of the second part is similar. The subspace not spanned by the columns of M_1 is stable if and only if the subspace not spanned by the columns of M_3 is stable. □

3.2 Numerical Algorithms

In this section we obtain necessary conditions for the optimal solution that we can numerically solve. We wish to find

$$F = \sum_{i=1}^{n} \alpha_i \Phi_i$$

such that trace$(PR(0))$ is minimized, where

$$P = (Q + F^T R F) + (A + BF)^T P (A + BF). \tag{2}$$

For a critical point

$$\text{trace}\left(\frac{\partial P}{\partial \alpha_i} R(0)\right) = 0, \qquad \forall i = 1, 2, \cdots, n.$$

Let us define

$$\Sigma_i = \Phi_i^T \left[RF + B^T P (A + BF)\right]. \tag{3}$$

Differentiating (2) with respect to α_i, we obtain

$$\frac{\partial P}{\partial \alpha_i} = (A + BF)^T \frac{\partial P}{\partial \alpha_i}(A + BF) + \Sigma_i + \Sigma_i^T.$$

Thus the cost is given by

$$\text{trace}\left(\frac{\partial P}{\partial \alpha_i} R(0)\right) = \text{trace}\left(\left((A + BF)^T \frac{\partial P}{\partial \alpha_i}(A + BF) + \Sigma_i + \Sigma_i^T\right) R(0)\right).$$

Now the covariance of the state at time k evolves as

$$R(k+1) = (A+BF)R(k)(A+BF)^T.$$

Thus

$$\text{trace}\left((A+BF)^T\frac{\partial P}{\partial \alpha_i}(A+BF)R(0)\right) = \text{trace}\left(\frac{\partial P}{\partial \alpha_i}R(1)\right).$$

Using this relation k times, we obtain

$$\text{trace}\left(\frac{\partial P}{\partial \alpha_i}R(0)\right) = \text{trace}\left(\left(\frac{\partial P}{\partial \alpha_i}R(k) + \Sigma_i X(k) + \Sigma_i^T X(k)\right)R(0)\right),$$

where

$$X(k) = R(0) + R(1) + \cdots + R(k).$$

But if $(A+BF)$ is stable, $R(k)$ would be approximately be a zero matrix for sufficiently large values of k. Thus if we denote

$$X = R(0) + R(1) + \cdots,$$

we see that X satisfies the Lyapunov equation

$$X = R(0) + (A+BF)X(A+BF)^T, \tag{4}$$

we obtain the following necessary condition for a critical point. We want

$$\text{trace}\left(\Sigma_i X + \Sigma_i^T X\right) = 0 \qquad\qquad \forall i = 1, \cdots, n,$$

where

$$F = \sum_{i=1}^{n} \alpha_i \Phi_i,$$

P satisfies (2), Σ_i is defined by (3) and X satisfies (4). This equation can either be solved iteratively or a gradient descent method can be used to obtain the control law.

As an exercise, show that

1. For the case when F has no restrictions on its structure, we obtain the usual condition
$$B^T P(A+BF) + RF = 0.$$

2. If the initial conditions of the agents are independent, then for the completely decentralized case (when the control law of each agent can depend only on its own state value), only the diagonal terms of cost matrices Q and R are important. Note that this is *not* true is general. Even if agent i cannot access the state of agent j, the (i, j)-th block of matrices Q and R are still important.

The algorithm we have discussed is for the infinite horizon case. For the finite horizon case, a similar algorithm can be applied as described, e.g., in [11]. However, as [12] pointed out, there are computational difficulties arising out of solving a large number of coupled matrix equations. A sub-optimal algorithm to get around this difficulty was proposed in [10] in which as opposed to NT coupled matrix equations (where T is the time horizon and N agents are present), N equations need to be solved T times.

3.3 Other Approaches

We have described one particular approach towards obtaining sub-optimal algorithms for the distributed control problem. Many other approaches have been proposed in the literature. We do not have time to go through them in any detail. However we summarize a couple of approaches here.

The problem of synthesizing a constrained controller while minimizing a H_2 performance criterion was considered in [13]. The analysis problem (for a given controller) was shown to be convex. However the it was shown that for the synthesis problem, enforcing the topology constraint typically destroys convexity. A method to retain convexity at the expense of sub-optimality was presented.

The problem of synthesizing a distributed controller achieving H_∞ performance was considered in [14]. They used tools inspired by dissipativity theory and derive sufficient LMI conditions on which performance constraints can be imposed. The controller structure that they come up with has the same interconnection topology as the plant interconnection topology. The tools have been extended to the case of lossy communication links in [15].

These are but two particular approaches. The problem can also be looked at in context of Receding Horizon Control. Distributed Receding Horizon Control will be covered in detail next week. There is also extensive work on many other approaches including those inspired by Game Theory, potential fields and so on.

4 Identifying Solvable Information Patterns

As we saw, the general problem of distributed control is very difficult and the optimal controller is not known for arbitrary information patterns. In particular, optimal controllers are not linear or even numerically easy to calculate in general. There have been numerous efforts to classify what information patterns lead to linear controllers being optimal and in what cases can the optimal linear controller be cast as a convex optimization problem. Witsenhausen [2] in a survey paper summarized several important results. He gave sufficient conditions under which the standard LQG theory could be applied and thus the optimal controller would be linear. Another important early contribution was [16] which showed the optimal controller to be linear for a class of information structures that they called *partially nested*. A partially nested information structure is one in which the *memory communication structure* is the same as the precedence relation in the *information structure diagram*. Informally, this means that a controller A has access to all the information that another controller B has access to, if the decision that B makes can affect the information set of A. Thus once the control laws are fixed, any controller can deduce the action of all the controllers *precedent* to it.The only random effects are due to the structure of the external disturbances which are not control-law dependent.

As an example, consider a system where two agents evolve according to

$$\begin{aligned} x_1(k+1) &= A_1x_1(k) + B_1u_1(k) + w_1(k) \\ x_2(k+1) &= A_2x_2(k) + A + 12x_1(k) + B_2u_2(k) + w_2(k), \end{aligned}$$

where $w_1(k)$ and $w_2(k)$ are white uncorrelated zero mean Gaussian noises. Further let the initial conditions $x_1(0)$ and $x_2(0)$ be independent. Suppose the cost function to be minimized is

$$J = E\left[\sum_{k=0}^{K}\{x_1^T(k+1)Q_1x_1(k+1) + x_2^T(k+1)Q_2x_2(k+1) + u_1^T(k)R_1u_1(k) + +u_2^T(k)R_2u_2(k)\}\right].$$

The agents are being observed through measurements of the form

$$\begin{aligned} y_1(k) &= C_1x_1(k) + v_1(k) \\ y_2(k) &= C_2x_2(k) + v_2(k), \end{aligned}$$

with the usual assumptions on the noises. Obviously if both the agents have access to all previous control inputs $u_i(0)$, $u_i(1)$, $\cdots$, $u_i(k-1)$ and the measurements $y_i(0)$, $y_i(1)$, $\cdots$, $y_i(k)$ at any time step k, the information

structure is classical. The problem then admits of unique optimal control inputs $u_i(k)$. Further they are linear in the measurements and can be obtained, e.g., using the LQG theory. However now consider an information pattern in which agent 1 has access to its own previous controls $u_1(0)$, $u_1(1)$, $\cdots$, $u_1(k-1)$ and its own measurements $y_1(0)$, $y_1(1)$, $\cdots$, $y_1(k)$. The agent 2 has access to its own control inputs but measurements from both agents. In this case, the information pattern is partially nested. Even for this information structure, the optimal control inputs are unique and linear in the measurements. This is so because agent 1 can choose its control input without worrying about agent 2's decision. Agent 2 can reconstruct agent 1's control input if it knows the control *law* followed by agent 1 even if it does not have access to the control *input* directly. Thus it can also solve for the same control input as in the classical information pattern case.

There has been a lot of work on particular information patterns. For instance the one-step delayed information sharing pattern assumes that each controller has, at the current time, all the previously implemented control values, all the observations made anywhere in the system through, and including the previous time, and its own observation at the current time. Hence current observations are not shared. Recursive solutions for this problem with a quadratic cost were provided using dynamic programming in [18], an exponential cost by [19] and with H_2, H_∞ and L_1 costs by [21]. Some other structures that are tractable have been identified, e.g., in [17, 20]. A property called *quadratic invariance* was defined in [22] and it was shown that it is necessary and sufficient for the constraint set to be preserved under feedback, and that this allows optimal stabilizing decentralized controllers to be synthesized via convex programming.

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