# Robust Control Theory: A New Approach

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## 1 Introduction

We consider $H_\infty$ analysis problem: the problem of computing $H_\infty$ norm (the $\ell_2$ induced gain from a bounded input to an output) of linear discrete time invariant system. The result from the robust control theory connects $H_\infty$ norm with the robust stability of a system in the presence of certain uncertainty [DP00]; The system with smaller $H_\infty$ norm is more robust to the uncertainty. Based on this result, the robust control theory aims to find a feedback controller which minimizes $H_\infty$ norm to maximize the robustness of the closed loop system, [Doy84], [DGKF89], and computing $H_\infty$ norm is the essential part of the robust control theory.

$H_\infty$ analysis is well studied in standard textbooks in control, [BEGFB94], [ZDG+96], and [DP00] to name a few. Moreover, numerical methods for $H_\infty$ analysis are also investigated in [BB90], [BS90], and [Sch90]. All the standard results are based on so called the KYP (Kalman-Yakubovich-Popov) lemma [Ran96]. Our recent work [YD13] points out that the KYP lemma based $H_\infty$ analysis can be seen as the Lagrangian dual of a well-defined optimization problem, and in fact this hidden duality is already investigated in [Sch06], [TL11b], and [Ebi09], to name a few.

Recently, [GB13] presented a new, simple approach to $H_\infty$ analysis by explicitly constructing the disturbance that (asymptotically) achieves $H_\infty$ norm of the system. Mathematically, the result from [GB13] is already known to be the Lagrangian dual of KYP lemma, but this approach suggests that KYP lemma may not be the only way to derive $H_\infty$ norm.

This report combines these new insights from [GB13] and [YD13] to reformulate the $H_\infty$ analysis problem. Specifically, we construct the disturbance that achieves the $H_\infty$ norm of the system, which is a sinusoidal signal, and this information gives us the exact frequency where the Bode magnitude plot has the maximum value. This construction is strikingly different from the KYP lemma where we are not able to recover the peak frequency nor the worst case disturbance.

Although our formulation is same as the one in [GB13], but we present alternative proof based on [YD13], since the original proof in [GB13] can't recover the frequency component of the disturbance. Moreover this new proof also generalizes towards bounded frequency $H_\infty$ analysis, where we consider a disturbance with bounded frequency components.

Finally, our result does not require controllability of the system, in contrast to the KYP lemma and [GB13] where we require controllability of the system. Therefore, this direct, and simple approach relaxes the requirement of the algorithm but also provides a way to construct the worst case disturbance, and we believe that this approach opens a new way of describing the robust control theory in modern optimization point of view.

## Notation

$W^*$ is the Hermitian of $W$, $\mathrm{Tr}\,(W)$ is a trace of $W$, and $W^\dagger$ is the pseudo inverse of $W$. The generalized inequality, $X \succ (\succeq)\, 0$, means $X$ is a positive (semi)definite matrix.

## 2 $H_\infty$ Analysis

### 2.1 Problem formulation

Let us formulate the $H_\infty$ analysis problem with discrete time dynamics. Consider LTI system $\mathcal{M}$:

$$\begin{aligned} x_{k+1} &= Ax_k + Bw_k \\ z_k &= Cx_k + Dw_k, \end{aligned}$$

$A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{l\times n}$, and $D \in \mathbb{C}^{l\times m}$. Moreover we assume that $A$ is Schur stable, the spectral radius of $A$ is less than unity. Although the $H_\infty$ analysis consider $\mathcal{L}_2$ gain of the system, it can be shown that this is equivalent to consider the power norm [ZGBD94]. Therefore for the ease of presentation, we proceed the analysis with the power norm.

Define the power norm of the signal $\mathbf{h}$ as

$$\|\mathbf{h}\|_p^2 = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} h_k^* h_k.$$

We define $H_\infty$ norm of the system $\mathcal{M}$ as:

$$\|\mathcal{M}\|_\infty = \sup_{\|\mathbf{w}\|_p \le 1} \frac{\|\mathbf{z}\|_p}{\|\mathbf{w}\|_p}.$$

From the linearity of the system, the supremum is always achieved at $\|\mathbf{w}\|_p = 1$. Therefore, for the linear system, above definition is equivalent to

$$\|\mathcal{M}\|_\infty = \sup_{\|\mathbf{w}\|_p \le 1} \|\mathbf{z}\|_p.$$

From the definition of the power norm, $\|\mathcal{M}\|_\infty^2$ can be calculated by the following infinite dimensional optimization problem:

$$\begin{aligned} \|\mathcal{M}\|_\infty^2 = \ \underset{\mathbf{w},\mathbf{x}}{\text{maximize}} \quad & \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} z_k^* z_k \\ \text{subject to} \quad & x_{k+1} = Ax_k + Bw_k \\ & z_k = Cx_k + Dw_k \\ & x_0 = 0 \\ & \|\mathbf{w}\|_p \le 1. \end{aligned} \tag{1}$$

Notice that this problem as posed is intractable since we have countably infinite number of variables and constraints. Without any reduction, we can't hope to solve this problem.

## 2.2 Main Result

Introduce the new variable

$$V = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \begin{bmatrix} x_k \\ w_k \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix}^* \succeq 0.$$

This lifting enables us to reduce the problem to a finite dimensional convex problem which we can solve.

First of all, the objective function only depends on this new matrix $V$:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} z_k^* z_k = \operatorname{Tr}\left( \begin{bmatrix} C & D \end{bmatrix} V \begin{bmatrix} C & D \end{bmatrix}^* \right).$$

From the dynamics $x_{k+1} = Ax_k + Bw_k$, we have $x_{k+1}x_{k+1}^* = (Ax_k + Bw_k)(Ax_k + Bw_k)^*$. By taking the infinite sum on both sides, we can conclude that

$$\begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} A^* \\ B^* \end{bmatrix},$$

and from $\|\mathbf{w}\|_p \leq 1$, we have

$$\mathrm{Tr}\left(\begin{bmatrix} 0 & I \end{bmatrix} V \begin{bmatrix} 0 & I \end{bmatrix}^*\right) \leq 1.$$

Notice that adding this redundant constraint to (1) does not change the optimal value and the solution of the problem.

$$\begin{array}{ll} \underset{V,\mathbf{x},\mathbf{w}}{\text{maximize}} & \mathrm{Tr}\left(\begin{bmatrix} C & D \end{bmatrix} V \begin{bmatrix} C & D \end{bmatrix}^*\right) \\ \text{subject to} & \begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} A^* \\ B^* \end{bmatrix} \\ & \mathrm{Tr}\left(\begin{bmatrix} 0 & I \end{bmatrix} V \begin{bmatrix} 0 & I \end{bmatrix}^*\right) \leq 1 \\ & V \succeq 0 \\ & x_{k+1} = Ax_k + Bw_k \\ & x_0 = 0 \\ & \|\mathbf{w}\|_p \leq 1 \\ & V = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \begin{bmatrix} x_k \\ w_k \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix}^* \end{array} \tag{2}$$

This is still problematic. First of all, (2) is still an infinite dimensional problem, and the last equality is not affine. Therefore this lifted problem (2) is an infinite-dimensional non-convex problem. However, by dropping the last four constraints, we have a relaxed version of (2) which is a finite dimensional semidefinite program [BV04]:

$$\begin{array}{rll} \mu_{\mathrm{opt}} = & \underset{V}{\text{maximize}} & \mathrm{Tr}\left(\begin{bmatrix} C & D \end{bmatrix} V \begin{bmatrix} C & D \end{bmatrix}^*\right) \\ & \text{subject to} & \begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} A^* \\ B^* \end{bmatrix} \\ & & \mathrm{Tr}\left(\begin{bmatrix} 0 & I \end{bmatrix} V \begin{bmatrix} 0 & I \end{bmatrix}^*\right) \leq 1 \\ & & V \succeq 0. \end{array} \tag{3}$$

One direct consequence of this relaxation is that the optimal value of (2) is less than (3), because (3) has larger feasible set, so the optimal value $\mu_{\mathrm{opt}}$ of (3) provides an upper bound of $\|\mathcal{M}\|_\infty^2$. However the following non-trivial result shows that $\mu_{\mathrm{opt}}$ is actually same as $\|\mathcal{M}\|_\infty^2$ and we can recover the optimal solution of (1) from the optimal solution $V_{\mathrm{opt}}$ of (3).

**Theorem 1.** *The optimal value of* (3)*,* $\mu_{opt}$*, is same as the optimal value of* (1)*,* $\|\mathcal{M}\|_\infty^2$*.*

Notice that, unlike [Ran96], [GB13], we do not require controllability of $(A, B)$ . Moreover, we will describe how to construct the optimal solution of (1), which is the problem of interest, from the optimal solution of (3), and this solution tells where the Bode magnitude plot has the maximum value. Before proving this result, we need some technical lemmas from Linear Algebra.

**Lemma 2** (Rank one decomposition)**.** *Suppose* $V \succeq 0$*, and* $\begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} A^* \\ B^* \end{bmatrix}$*. Then there exists a set of rank one matrices* $V_k$ *such that*

$$V = \sum_k V_k, \ \text{Rank}(V_k) = 1, \ V_k \succeq 0$$
$$\begin{bmatrix} I & 0 \end{bmatrix} V_k \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} V_k \begin{bmatrix} A^* \\ B^* \end{bmatrix}$$

*Proof.* The following proof is from [Ran96]. Let $F = \begin{bmatrix} I & 0 \end{bmatrix} V^{1/2}$, and $G = \begin{bmatrix} A & B \end{bmatrix} V^{1/2}$. Since $FF^* = GG^*$, Lemma 7 in the appendix implies that there exists a unitary matrix $U$ such that $F = GU$. Being unitary, $U = \sum_k e^{j\theta_k} u_k u_k^*$, and $\sum_k u_k u_k^* = I$. Notice that $Fu_k = GUu_k = e^{j\theta_k} G u_k$, and $F u_k u_k^* F^* = G u_k u_k^* G^*$. Therefore, by defining $V_k = V^{1/2} u_k u_k^* V^{1/2}$, it is routine to check that this construction satisfies all the constraints. □

The above lemma shows that the extreme points of the feasible set of (3) are rank one matrices. Since the objective function in (3), $\text{Tr}\left(\begin{bmatrix} C & D \end{bmatrix} V \begin{bmatrix} C & D \end{bmatrix}^*\right)$, is *linear*, there exists a rank one optimal solution of (3).

**Proposition 3** (Rank one optimal solution)**.** *There exists an optimal solution of* (3)*,* $V_{opt}$ *with* $\text{Rank}(V_{opt}) = 1$.

*Proof.* Let $V_{\text{opt}}$, and $\mu_{\text{opt}}$ be the optimal solution, and the optimal value of (3), respectively. From Lemma 2, there exist rank one matrices $V_k$ such that $V_{\text{opt}} = \sum_k V_k$ and $V_k$ is in the feasible set of (3).

Define the scalar values:

$$\begin{aligned} p_k &= \text{Tr}\left(\begin{bmatrix} 0 & I \end{bmatrix} V_k \begin{bmatrix} 0 & I \end{bmatrix}^*\right), \\ \mu_k &= \text{Tr}\left(\begin{bmatrix} C & D \end{bmatrix} V_k \begin{bmatrix} C & D \end{bmatrix}^*\right). \end{aligned}$$

Then, we must have $\mu_{\text{opt}} = \sum_k \mu_k$, and $\sum_k p_k \leq 1$. Let $J$ be the index such that

$$J = \underset{k}{\operatorname{argmax}} \frac{\mu_k}{p_k}.$$

(Notice that from Proposition 12 in the appendix, $p_k > 0$, for all $k$.)

We will show that $\hat{V} = \frac{1}{p_J} V_J$ is a rank one optimal solution of (3). It is easy to check that $\hat{V}$ is a feasible point of (3). Moreover,

$$\begin{aligned} \operatorname{Tr}\left(\begin{bmatrix} C & D \end{bmatrix} \hat{V} \begin{bmatrix} C & D \end{bmatrix}^*\right) &= \frac{\mu_J}{p_J} \\ &\geq \sum_k p_k \frac{\mu_J}{p_J} \\ &\geq \sum_k p_k \frac{\mu_k}{p_k} \\ &= \sum_k \mu_k = \mu_{\text{opt}}. \end{aligned}$$

Therefore, $\operatorname{Tr}\left(\begin{bmatrix} C & D \end{bmatrix} \hat{V} \begin{bmatrix} C & D \end{bmatrix}^*\right) \geq \mu_{\text{opt}}$ which implies $\operatorname{Tr}\left(\begin{bmatrix} C & D \end{bmatrix} \hat{V} \begin{bmatrix} C & D \end{bmatrix}^*\right) = \mu_{\text{opt}}$, and $\hat{V}$ is a rank one optimal solution. □

Now we are ready to prove Theorem 1.

*Proof of Theorem 1.* From Proposition 3, we obtain a rank one optimal solution $V_{\text{opt}}$ of (3). Since $\operatorname{Rank}(V_{\text{opt}})$ is one, there exist vectors $x_{\text{opt}} \in \mathbb{C}^n$, and $w_{\text{opt}} \in \mathbb{C}^m$ such that $V_{\text{opt}} = \begin{bmatrix} x_{\text{opt}} \\ w_{\text{opt}} \end{bmatrix} \begin{bmatrix} x_{\text{opt}} \\ w_{\text{opt}} \end{bmatrix}^*$. Since $V_{\text{opt}}$ satisfies (4), we have

$$\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} x_{\text{opt}} \\ w_{\text{opt}} \end{bmatrix} \begin{bmatrix} x_{\text{opt}} \\ w_{\text{opt}} \end{bmatrix}^* \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x_{\text{opt}} \\ w_{\text{opt}} \end{bmatrix} \begin{bmatrix} x_{\text{opt}} \\ w_{\text{opt}} \end{bmatrix}^* \begin{bmatrix} A^* \\ B^* \end{bmatrix},$$

and from Corollary 8 in the appendix, there exists a scalar $\theta_{\text{opt}}$ such that

$$e^{j\theta_{\text{opt}}} x_{\text{opt}} = A x_{\text{opt}} + B w_{\text{opt}}.$$

Therefore, $w_k = e^{j\theta_{\text{opt}} k} w_{\text{opt}}$ results in $x_k = (e^{j\theta_{\text{opt}} k} I_n - A^k) x_{\text{opt}}$, and since $A$ is stable, we have

$$V_{\text{opt}} = \lim_{n \to \infty} \frac{1}{n} \begin{bmatrix} x_k \\ w_k \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix}^*.$$

From this point, it is obvious that $\|\mathbf{w}\|_p \leq 1$. In fact $\|\mathbf{w}\|_p = 1$, and $\|\mathbf{z}\|_p = \mu$. Therefore, $\frac{\|\mathbf{z}\|_p}{\|\mathbf{w}\|_p} = \mu$, which concludes the proof. □

**Remark 1.** In the proof, we construct $\mathbf{w}$, the optimal solution of (1). This worst case disturbance is a sinusoid which is well known in the literature, since the $H_\infty$ norm of the system is the maximum value in the Bode magnitude plot. However, in contrast to other approach, *e.g.* [Ran96], we explicitly construct the disturbance $\mathbf{w}$, and its frequency component $\theta_{\mathrm{opt}}$.

### 2.3 The worst case disturbance extraction

As Remark 1 points out, we can construct the worst case signal by solving (3) based on our new proof. Let us describe the detail procedure.

Suppose we obtain a solution $V_{\mathrm{opt}}$ of (3). If $V_{\mathrm{opt}}$ is rank one, then it requires no additional step. Simply find a pair of vectors $(x_{\mathrm{opt}}, w_{\mathrm{opt}})$, $V_{\mathrm{opt}} = \begin{bmatrix} x_{\mathrm{opt}} \\ w_{\mathrm{opt}} \end{bmatrix} \begin{bmatrix} x_{\mathrm{opt}} \\ w_{\mathrm{opt}} \end{bmatrix}^*$, then $\theta_{\mathrm{opt}}$ is guaranteed to exists such that $e^{j\theta_{\mathrm{opt}}} x_{\mathrm{opt}} = Ax_{\mathrm{opt}} + Bw_{\mathrm{opt}}$. Therefore we can use element-wise devision among $x_{\mathrm{opt}}$ and $Ax_{\mathrm{opt}} + Bw_{\mathrm{opt}}$ to find the frequency $\theta_{\mathrm{opt}}$. From here the worst case disturbance is $w_k = e^{j\theta_{\mathrm{opt}} k} w_{\mathrm{opt}}$. If $V_{\mathrm{opt}}$ is not rank one, we can use the procedure in the proof of Proposition 3 to recover rank one solution, then apply aforementioned procedure. To do this we need a unitary matrix $U$ which satisfies

$$\begin{bmatrix} I & 0 \end{bmatrix} V_{\mathrm{opt}}^{1/2} = \begin{bmatrix} A & B \end{bmatrix} V_{\mathrm{opt}}^{1/2} U.$$

The following algorithm from [IMF00] constructs a desired $U$. The correctness of this algorithm can be proved using argument in [IMF00], so we omit the detailed proof.

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**Algorithm 1**
**Input:** Complex matrices, $F, G$ such that $FF^* = GG^*$
**Output:** A unitary matrix $U$ such that $F = GU$

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1. Set $P = F + G$, and $Q = F - G$
2. Find the SVD of $P = U_P \Sigma_P V_P^*$, and let $r = \mathrm{Rank}(P)$
3. Set $\begin{bmatrix} R & S \end{bmatrix} = \begin{bmatrix} I_r & 0 \end{bmatrix} V_P^* P^\dagger Q V_P$
4. Set $\Delta = V_P \begin{bmatrix} R & S \\ -S^* & 0 \end{bmatrix} V_P^*$
5. $U = (I + \Delta)(I - \Delta)^{-1}$

By applying Algorithm 1 to $F = \begin{bmatrix} I & 0 \end{bmatrix} V_{\text{opt}}^{1/2}$, and $G = \begin{bmatrix} A & B \end{bmatrix} V_{\text{opt}}^{1/2}$, we obtain a desired unitary matrix $U$ in the proof of Lemma 2. The second step is to perform eigenvalue decomposition of $U$ to have $U = \sum_k e^{j\theta_k} u_k u_k^*$, where $u_k$ is the eigenvector of $U$. The third step is to find $V_k = V_{\text{opt}}^{1/2} u_k u_k^* V_{\text{opt}}^{1/2}$. The final step is to find a index $J$ which maximizes $\frac{\mu_k}{p_k}$ as in the Proposition 3. Then $V_J$ is a rank one optimal solution.

### 2.4 Connection to the KYP lemma

Textbooks in the $H_\infty$ control, for example [DP00], use the KYP lemma combined with bisection search to obtain the $H_\infty$ norm of the system. In fact, the Lagrangian dual of our optimization (3) generates the optimization derived from the KYP lemma:

$$\begin{aligned} \underset{\lambda, P}{\text{minimize}} \quad & \lambda \\ \text{subject to} \quad & \begin{bmatrix} A^*PA - P & A^*PB \\ B^*PA & B^*PB \end{bmatrix} + \begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D - \lambda I \end{bmatrix} \preceq 0 \\ & \lambda \geq 0, P = P^*. \end{aligned} \tag{4}$$

However there is no guarantee on the optimal value from (4) to be the same as $H_\infty$ norm of the system, because strong duality may not hold. The following example shows the case where strong duality fails.

**Example 1.** *Consider the scalar system with $(A, B, C, D) = (0, 0, 1, 1)$. The optimal value of (3) is 1, and so as $\|\mathcal{M}\|_\infty = 1$, whereas the optimal value of (4) is $+\infty$ because there is no feasible point.*

To establish strong duality, we need controllability:

**Proposition 4.** *Suppose $(A, B)$ is controllable. Then, strong duality holds between (3), and (4).*

The basic idea is to construct a positive definite feasible point $V \succ 0$ in (3), and use Slater's condition as in [YD13]. Since the construction is very technical, we relegate the proof in the appendix.

Proposition (4) suggests that the controllability requirement in the KYP lemma is from this duality issue. In contrast, our direct approach (3) does not require controllability which is another benefit.

## 3 Bounded Frequency $H_\infty$ Analysis

As we have seen, the input that achieves the $H_\infty$ norm is a sinusoidal signal with frequency that can be arbitrary in the interval $[-\pi, \pi]$. In this section, we consider disturbances with bounded frequency. Specifically, for a given frequency $0 < \theta_0 < \pi$, we consider the disturbance with the specific form: $\mathcal{W}_L = \{\mathbf{w} : w_k = e^{j\theta k} w_s,\ \theta \in [-\theta_0, \theta_0]\}$. In other words, the support of the (discrete time) Fourier transform of $\mathbf{w}$ is contained by $[-\theta_0, \theta_0]$. This formulation seeks the maximum value of the Bode magnitude plot of the system in the low frequency region, $[-\theta_0, \theta_0]$, not in the entire region, $[-\pi, \pi]$:

$$\begin{array}{ll} \underset{\mathbf{w},\mathbf{x}}{\text{maximize}} & \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} z_k^* z_k \\ \text{subject to} & x_{k+1} = Ax_k + Bw_k \\ & z_k = Cx_k + Dw_k \\ & x_0 = 0 \\ & \|\mathbf{w}\|_p \le 1 \\ & \mathbf{w} \in \mathcal{W}_L. \end{array} \tag{5}$$

Because of the frequency constraint on $\mathbf{w}$, it is not obvious how to lift this problem as in the $H_\infty$ analysis, but we show that this can be also casted as a semidefinite program.

Consider the disturbance $w_k = e^{j\theta k} w_s \in \mathcal{W}_L$. This results in $x_k = e^{j\theta k} x_s + x_h$, where

$$e^{j\theta} x_s = Ax_s + Bw_s,$$

and $x_h = -A^k x_s$ is a transient term which goes to zero asymptotically. Notice that from the dynamics, we have that

$$x_{k+1}x_k^* + x_k x_{k+1}^* = (Ax_k + Bw_k)x_k^* + x_k(Ax_k + Bw_k)^*,$$

and from the solution of dynamics, $x_k = e^{j\theta k} x_s + h_k$, where $h_k = -A^k x_s$,we have

$$\begin{aligned} x_{k+1}x_k^* + x_k x_{k+1}^* \quad &= \quad 2\cos\theta x_s x_s^* + e^{j\theta k}(1 + e^{j\theta}) x_s h_k^* \\ &\quad\; + e^{-j\theta k}(1 + e^{-j\theta}) h_k x_s^* + h_k h_k^*. \end{aligned}$$

Let us define $V = \lim_{n\to\infty} \frac{1}{n} \begin{bmatrix} x_k \\ w_k \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix}^*$. Since $h_k$ converges to zero exponentially due to the stability of $A$, taking infinite sum in the above two equations gives

$$2\cos\theta \begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} I \\ 0 \end{bmatrix} + \begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} A^* \\ B^* \end{bmatrix}.$$

and since $\theta \in [-\theta_0, \theta_0]$, we have $\cos\theta \geq \cos\theta_0$. Therefore

$$2\cos\theta_0 \begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} I \\ 0 \end{bmatrix} \preceq \begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} I \\ 0 \end{bmatrix} + \begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} A^* \\ B^* \end{bmatrix}.$$

Similar approach in $H_\infty$ analysis allows us to have,

$$\begin{aligned} \underset{V}{\text{maximize}} \quad & \operatorname{Tr}\left(\begin{bmatrix} C & D \end{bmatrix} V \begin{bmatrix} C & D \end{bmatrix}^*\right) \\ \text{subject to} \quad & \begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} A^* \\ B^* \end{bmatrix} \\ & \begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} I \\ 0 \end{bmatrix} + \begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} A^* \\ B^* \end{bmatrix} \succeq 2\cos\theta_0 \begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} I \\ 0 \end{bmatrix} \\ & \operatorname{Tr}\left(\begin{bmatrix} 0 & I \end{bmatrix} V \begin{bmatrix} 0 & I \end{bmatrix}^*\right) \leq 1 \\ & V \succeq 0. \end{aligned} \tag{6}$$

Since this is a relaxed version of (5), the optimal value of (6) yields an upper bound of (5). In fact, this gap is zero as in Theorem 1.

**Proposition 5.** *The optimal value of* (6)*equals the optimal value of* (5)*.*

The proof is almost identical once we have a rank one decomposition of $V_{\text{opt}}$. For notational simplicity, let $\mathcal{F}_L$ be the feasible set of (6).

**Lemma 6** (Rank one decomposition)**.** *For all $V \in \mathcal{F}_L$, there exists a set of rank one matrices, $V_k \in \mathcal{F}_L$ such that,*

$$V = \sum_k V_k, \ \operatorname{Rank}(V_k) = 1.$$

*Proof.* Define $F = \begin{bmatrix} I & 0 \end{bmatrix} V^{1/2}$, and $G = \begin{bmatrix} A & B \end{bmatrix} V^{1/2}$. Then from Lemma 10 in the appendix, there exists a unitary matrix $U$ such that $F = GU$, and $U + U^* \succeq 2\cos\theta I$.

Being unitary, $U = \sum_k e^{j\theta_k} u_k u_k^*$. By defining $V_k = V^{1/2} u_k u_k^* V^{1/2}$, we can easily check that $V_k \in \mathcal{F}_L$, and $V = \sum_k V_k$. □

To extract a maximizing input, we can use exactly same procedure as in Section II. C except finding a unitary matrix $U$ due to additional requirement $U + U^* \succeq 2\cos\theta I$. Following algorithm from [IMF00] finds a desired $U$.

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**Algorithm 2**
**Input:** Complex matrices, $F, G$ such that $FF^* = GG^*$, $FG^* + GF^* \succeq 2\cos\theta I$.
**Output:** A unitary matrix $U$ such that $F = GU$, and $U + U^* \succeq 2\cos\theta I$.

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1. Set $\mu = \frac{1-\cos\theta}{1+\cos\theta}$.
2. Set $P = \sqrt{\mu}(F+G)$, and $Q = F - G$.
3. Find the SVD of $P = U_P \Sigma_P V_P^*$, and let $r = \text{Rank}(P)$.
4. Set $\begin{bmatrix} R & S \end{bmatrix} = \begin{bmatrix} I_r & 0 \end{bmatrix} V_P^* P^\dagger Q V_P$.
5. Set $\Delta = V_P \begin{bmatrix} R & S \\ -S^* & -S^* R (I_r + R^2)^\dagger S \end{bmatrix} V_P^*$.
6. $U = (I + \sqrt{\mu}\Delta)(I - \sqrt{\mu}\Delta)^{-1}$.

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Also, notice that from Lemma 10, $\theta_{\text{opt}}$ is guaranteed to be in $[-\theta_0, \theta_0]$.

### 3.1 Connection to Generalized KYP

Following is the Lagrangian dual problem of (6):

$$\begin{aligned} \underset{\lambda, P}{\text{minimize}} \quad & \lambda \\ \text{subject to} \quad & \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} P & Q \\ Q & -P - 2\cos\theta_0 Q \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \\ & + \begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix} \preceq \lambda \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \\ & \lambda \geq 0, P = P^*, Q \succeq 0, \end{aligned} \tag{7}$$

and this can be also derived from the Generalized KYP lemma [IH05]. However, as in the $H_\infty$ analysis, the strong duality issue also arises.

**Example 2.** *Consider the scalar system with $(A, B, C, D) = (0, 0, 1, 1)$. The optimal value of (6) is 1, and so as $\|\mathcal{M}\|_\infty = 1$ over the low frequency region $[-\theta_0, \theta_0]$, whereas the optimal value of (7) is $+\infty$ because there is no feasible point.*

One condition for strong duality is controllability of $(A, B)$. Since the proof is identical with Proposition 4, we omit the details.

### 3.2 $H_\infty$ Analysis with High Frequency Disturbance

So far we only considered the low frequency input. Using similar argument, as in the low frequency input case, we can also include the cases with high frequency disturbances, $\mathcal{W}_H = \{\mathbf{w} : w_k = e^{j\theta k} w_s, \theta \in [-\pi, -\theta_0] \cup [\theta_0, \pi]\}$, or the middle frequency disturbances, $\mathcal{W}_M = \{\mathbf{w} : w_k = e^{j\theta k} w_s, \theta \in [\theta_1, \theta_2]\}$. For high frequency disturbances, $\mathcal{W}_H$, a similar approach in the low frequency input gives us

$$\begin{aligned}
\underset{V}{\text{maximize}} \quad & \mathrm{Tr}\left(\begin{bmatrix} C & D \end{bmatrix} V \begin{bmatrix} C & D \end{bmatrix}^*\right) \\
\text{subject to} \quad & \begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} A^* \\ B^* \end{bmatrix} \\
& \begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} I \\ 0 \end{bmatrix} + \begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} A^* \\ B^* \end{bmatrix} \preceq 2\cos\theta_0 \begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} I \\ 0 \end{bmatrix} \\
& \mathrm{Tr}\left(\begin{bmatrix} 0 & I \end{bmatrix} V \begin{bmatrix} 0 & I \end{bmatrix}^*\right) \leq 1 \\
& V \succeq 0.
\end{aligned} \tag{8}$$

For middle frequency disturbances, $[\theta_1, \theta_2]$, we can use (6) by shifting $B, D$. Define $\theta_c = \frac{1}{2}(\theta_1 + \theta_2)$, and $\theta_0 = \frac{1}{2}(\theta_2 - \theta_1)$. Then the spectrum of $\mathbf{w}$ confined in $[\theta_1, \theta_2]$ is equivalent to that the new input $\tilde{\mathbf{w}} = e^{-j\theta_c}\mathbf{w}$ has finite spectrum on $[-\theta_0, \theta_0]$. In this coordinate, $B\mathbf{w} = Be^{j\theta_c}\tilde{\mathbf{w}}$, and $D\mathbf{w} = De^{j\theta_c}\tilde{\mathbf{w}}$. Therefore by defining $\tilde{B} = Be^{j\theta_c}$, and $\tilde{D} = De^{j\theta_c}$, we have the following optimization problem:

$$\begin{aligned}
\underset{V}{\text{maximize}} \quad & \mathrm{Tr}\left(\begin{bmatrix} C & \tilde{D} \end{bmatrix} V \begin{bmatrix} C & \tilde{D} \end{bmatrix}^*\right) \\
\text{subject to} \quad & \begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} A & \tilde{B} \end{bmatrix} V \begin{bmatrix} A^* \\ \tilde{B}^* \end{bmatrix} \\
& \begin{bmatrix} A & \tilde{B} \end{bmatrix} V \begin{bmatrix} I \\ 0 \end{bmatrix} + \begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} A^* \\ \tilde{B}^* \end{bmatrix} \succeq 2\cos\theta_0 \begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} I \\ 0 \end{bmatrix} \\
& \mathrm{Tr}\left(\begin{bmatrix} 0 & I \end{bmatrix} V \begin{bmatrix} 0 & I \end{bmatrix}^*\right) \leq 1 \\
& V \succeq 0.
\end{aligned} \tag{9}$$

We can also derive dual problems which also can be derived from the Generalized KYP lemma. However, since the derivation is similar, we omit the details.

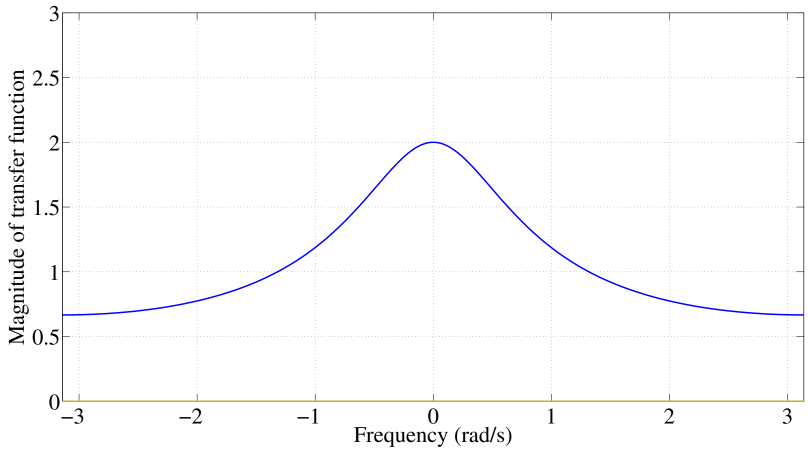


Figure 1: Bode magnitude plot of the scalar system $(A, B, C, D) = (\frac{1}{2}, \frac{1}{2}, 1, 1)$.

# 4 Numerical Example

We consider the example in [GB13], $(A, B, C, D) = (\frac{1}{2}, \frac{1}{2}, 1, 1)$. For Bode magnitude plot, $\|C(e^{j\omega}I - A)^{-1}B + D\|$, see Fig. 1. Firstly we solve (3), and the optimal solution is $V_{\text{opt}} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Notice that $V_{\text{opt}}$ is rank one, and applying our procedure to extract worst case disturbance, we get $x_{\text{opt}} = 1$, and $w_{\text{opt}} = 1$. To calculate the peak frequency, we need to solve $e^{j\theta_{\text{opt}}} x_{\text{opt}} = Ax_{\text{opt}} + Bw_{\text{opt}}$. Obviously, $e^{j\theta_{\text{opt}}} = 1$, and $\theta_{\text{opt}} = 0$. This is consistent with the Bode magnitude plot, Fig. 1, where $H_\infty$ norm is attained when $\theta = 0$.

To consider bounded frequency disturbance, we set $\theta_0 = \frac{\pi}{4}$, and solve (6) and (8) for low, and high frequency disturbance, respectively. From (6), we obtain the optimal solution $V_L = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. This makes sense because among the frequency $[-\frac{\pi}{4}, \frac{\pi}{4}]$, $\theta = 0$ achieves the peak value. Therefore, we should expect the same optimal value and optimal solution as in (3).

However, for high frequency disturbance, (8), we obtain the optimal solution $V_H = \begin{bmatrix} 0.4605 & 0.1907 \\ 0.1907 & 1 \end{bmatrix}$ of which rank is not one. By applying our procedure we obtain, rank one matrices, $V_1 = \begin{bmatrix} 0.2302 & 0.0954 + 0.3256i \\ 0.0954 - 0.3256i & 0.5 \end{bmatrix}$, and $V_2 = V_1^*$. Notice that $V = V_1 + V_2$, and it can be easily check that $V_1, V_2$

are in the feasible set and $V_1, V_2$ have same objective value as $V_H$. Therefore, $V_1$ and $V_2$ are rank one optimal solutions. Finally, by decomposing $V_1$, we obtain $x_L = 0.1907 - 0.6513i$, and $w_L = 1$. From $e^{j\theta} x_L = Ax_L + Bw_L$, we have $e^{j\theta} = e^{j\frac{\pi}{4}}$. Therefore, the worst case disturbance is $w_k = e^{j\frac{\pi}{4}k}$, and this is consistent with Fig. 1, where the peak occurs at $\frac{\pi}{4}$ if we consider the region $[-\pi, -\pi/4] \cup [\pi/4, \pi]$.

# 5 Conclusion

In this paper we proposed a simple, direct approach to $H_\infty$ analysis. Compared to the classical approach based on the KYP lemma, our approach can construct an explicit disturbance that achieves the $H_\infty$ norm of the system, and does not require the controllability condition of the system to calculate the $H_\infty$ norm. Moreover, we generalize this approach to the low, middle and high frequency disturbances and show the effectiveness of our new approach.

# 6 Proposed Research

## 6.1 Specialized solver for $H_\infty$ analysis

Although we have a semidefinite program for $H_\infty$ analysis, we can also exploit special structures in our optimization problem. Custom solvers for $H_\infty$ analysis with KYP lemma, which is the Lagrangian dual of our primal problem, are developed in [BB90], [Par99], and [LV07]. This suggests that our primal problem also have very special structure. Therefore, it would be a natural direction to connect existing works in the dual problem to our primal problem.

## 6.2 Scalable $H_\infty$ analysis

$H_\infty$ analysis problem has a linear matrix constraint $V \succeq 0$. Generic software such as SDPT3 [TTT99], or SeDuMi [Stu99] can solve this problem numerically, but in practice it can handle only unto moderate size of problem ($50 \times 50$ matrices). To handle larger system, we have to find numerically scalable algorithm. Recently, [MT] proposes a scalable sum of squares method where the author replaces a positive semidefinite matrix by a diagonally dominant matrix which results in a linear program and/or a matrix with all $2 \times 2$ subblocks being positive semidefinite which results in a second order cone program. Although this relaxation offers a lower bound of the problem, as a result we have a scalable algorithm since a linear program and a second order cone program is computationally cheaper than a semidefinite program. We can apply this idea to have a linear/second order cone program formulation of $H_\infty$ to have a scalable algorithm which offers a lower bound of the true $H_\infty$ norm.

Moreover, recent result in [TL11a] shows that, for some class of system, the computation complexity of $H_\infty$ analysis with KYP lemma can be dramatically reduced. This suggests that it may be true that for a certain class of system, our linear/second order cone relaxation offers the exact $H_\infty$ norm. Therefore it is worth to investigate when this relaxation is tight.

### 6.3 Towards $H_\infty$ controller design

Standard approach to $H_\infty$ controller synthesis is based on the KYP lemma. Therefore, it is natural to come up with $H_\infty$ synthesis with our direct formulation. For example, consider a static state feedback controller $u = Kx$. In this case, $H_\infty$ optimal controller is given by:

$$\begin{array}{lll} \underset{K}{\text{minimize}} & \underset{V}{\text{maximize}} & \mathrm{Tr}\left(\begin{bmatrix} C + D_2K & D_1 \end{bmatrix} V \begin{bmatrix} C + D_2K & D_1 \end{bmatrix}^*\right) \\ \text{subject to} & & \begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} A + B_2K & B_1 \end{bmatrix} V \begin{bmatrix} (A + B_2K)^* \\ B_1^* \end{bmatrix} \\ & & \mathrm{Tr}\left(\begin{bmatrix} 0 & I \end{bmatrix} V \begin{bmatrix} 0 & I \end{bmatrix}^*\right) \le 1 \\ & & V \succeq 0. \end{array} \tag{10}$$

In general, this problem is non-convex, but the minimax structure of the problem suggests that the minimax iteration may be a good heuristic for the problem. Notice that we can naturally encode structural constraint on $K$ to have distributed controller $K$ in the minimization step. This minimax nature of the problem should be exploited to have a better $H_\infty$ synthesis theory.

# 7 Appendix

### 7.1 Results from Linear Algebra

**Lemma 7** (A. Rantzer, 1996). *Let $F, G \in \mathbb{C}^{n\times m}$. The following statements are equivalent.*
*(i) $FF^* = GG^*$.*
*(ii) There exists a unitary matrix $U$ such that, $F = GU$.*

*Proof.* See [Ran96]. □

For a special case of Lemma 7, consider $f, g \in \mathbb{C}^{n\times 1}$. In this case, a unitary matrix $U$ is actually scalar, and we get following immediate consequence.

**Corollary 8.** *$ff^* = gg^*$ if and only if $f = e^{j\theta}g$ for some $\theta$.*

**Lemma 9** (T. Iwasaki, 2000)**.** *The following statements are equivalent.*
*(i) $FF^* \preceq GG^*$ and $FG^* + GF^* = 0$.*
*(ii) There exists a skew-symmetric matrix $\Delta = -\Delta^*$ such that, $F = G\Delta, ||\Delta|| \leq 1$.*

*Proof.* See [Ebi09] or [IMF00] □

The next result is a consequence of Lemma 9.

**Lemma 10.** *The following statements are equivalent.*
*(i) $FF^* = GG^*$, and $FG^* + GF^* \succeq 2\cos\theta FF^*$.*
*(ii) There exists a unitary matrix $U$ such that, $F = GU$ and, $U + U^* \succeq 2\cos\theta I$.*

*Proof.* From the direction (ii) to (i) is trivial. Let us show the direction from (i) to (ii).

Define $\mu = \frac{1-\cos\theta}{1+\cos\theta} < 1$, $P = F - G$, and $Q = \sqrt{\mu}(F + G)$. Then, (i) is equivalent to

$$PQ^* + QP^* = 0, \qquad PP^* \preceq QQ^*.$$

From Lemma 9, there exists a matrix $\Delta$ such that

$$P = Q\Delta, \qquad ||\Delta|| \leq 1, \qquad \Delta + \Delta^* = 0.$$

Since $\Delta$ is skew-symmetric, we can find a unitary matrix $S$ such that

$$\Delta = S\text{diag}\{j\lambda_i\}S^*,$$

where $j\lambda_i$ is the $i$th eigenvalue of $\Delta$. From the condition $||\Delta|| \leq 1$, we have $|\lambda_i| \leq 1$.

Now let us define $U = S\ \text{diag}\{\frac{1+j\sqrt{\mu}\lambda_i}{1-j\sqrt{\mu}\lambda_i}\}S^*$. Notice that $U^* = S\ \text{diag}\{\frac{1-j\sqrt{\mu}\lambda_i}{1+j\sqrt{\mu}\lambda_i}\}S^*$. Then, it is obvious that $UU^* = I$. Notice that $\frac{1+j\sqrt{\mu}\lambda_i}{1-j\sqrt{\mu}\lambda_i} + \frac{1-j\sqrt{\mu}\lambda_i}{1+j\sqrt{\mu}\lambda_i} = 2\frac{1-\mu\lambda_i^2}{1+\mu\lambda_i^2}$, and since $\lambda_i^2 \leq 1$, we have $\frac{1-\mu\lambda_i^2}{1+\mu\lambda_i^2} \geq \frac{1-\mu}{1+\mu}$. Therefore,

$$\begin{aligned} U + U^* &= S \operatorname{diag}\{2\frac{1-\mu\lambda_i^2}{1+\mu\lambda_i^2}\}S^* \\ &\succeq 2\frac{1-\mu}{1+\mu}SS^* = 2\cos\theta I, \end{aligned}$$

which concludes the proof. □

Again, for a special case of Lemma 10, consider $f, g \in \mathbb{C}^{n\times 1}$. We get following immediate consequence.

**Corollary 11.** *$ff^* = gg^*$, $fg^* + gf^* \preceq 2\cos\theta_0 ff^*$, if and only if $f = e^{j\theta}g$ for some $\theta \in [-\theta_0, \theta_0]$.*

## 7.2 Removing Singularity

**Proposition 12.** *Let $A$ be Schur stable. Consider $V \succeq 0$ such that*

$$\begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} A^* \\ B^* \end{bmatrix}, \text{ and } Rank(V) = 1,$$

*then, $Tr\left(\begin{bmatrix} 0 & I \end{bmatrix} V \begin{bmatrix} 0 & I \end{bmatrix}^*\right) > 0$.*

*Proof.* From the decomposition, $V = \begin{bmatrix} x \\ w \end{bmatrix}\begin{bmatrix} x \\ w \end{bmatrix}^*$, if $\mathrm{Tr}\left(\begin{bmatrix} 0 & I \end{bmatrix} V \begin{bmatrix} 0 & I \end{bmatrix}^*\right) = 0$, then $w = 0$. Moreover from Corollary 8, there exists $e^{j\theta}$ such that $e^{j\theta}x = Ax + Bw = Ax$, since $w = 0$. However, since $A$ is Schur stable, $e^{j\theta}I - A$ is invertible, therefore $x = 0$. This implies $V = 0$ which is contradict to $\mathrm{Rank}(V) = 1$. □

## 7.3 Proof of Proposition 4

*Proof.* Now we find a basis $\{v_i\}$ for $\mathbb{C}^{n+m}$ to construct

$$V = \sum_{i=1}^{n+m} v_i v_i^*,$$

which satisfies the strict inequality constraints in (3). Then the Slater's constraint qualification gives strong duality. Pick $n+1$ numbers on the unit disk $e^{j\theta_0}, e^{j\theta_1}, \cdots, e^{j\theta_n}$, that are distinct[1].

[1] $\theta_i$ can be chosen in the specific frequency region to generalize the proof to bounded frequency $H_\infty$ analysis case

Since $(A, B)$ is controllable, there exists a matrix $K$ such that the eigenvalues of $A - BK$ are $e^{j\theta_1}, \cdots, e^{j\theta_n}$. Denote corresponding eigenvector $x_i$,

$$(A - BK)x_i = e^{j\theta_i} x_i. \tag{11}$$

for $i = 1, \cdots, n$.

The rank of a matrix $T = \begin{bmatrix} A - e^{j\theta_0} I & B \end{bmatrix}$ is $n$, because of the Popov-Belevitch-Hautus (PBH) controllability test [DP00]. Therefore there exists a basis $\{t_1, \cdots, t_m\}$ for $N(T)$, the null space of $T$, and by substitution, we can show that $t_i t_i^*$ is a feasible point of problem (3), for all $i = 1, \cdots, m$.

Define $v_{n+k} = t_k$, and let $S_1 = \text{span}(v_1, , v_n)$, and $S_2 = N(T)$. Suppose,

$$v = \begin{bmatrix} x \\ u \end{bmatrix} \in S_1 \cap S_2.$$

Since $v \in S_1$, there exists $\{\alpha_i\}_{i=1}^n$ such that $v = \sum_{i=1}^n \alpha_i v_i$, which implies $x = \sum_{i=1}^n \alpha_i x_i$, and $u = \sum_{i=1}^n \alpha_i w_i$. Furthermore, $e^{j\theta_0} x = Ax + Bu$, because $v \in S_2$. Combining these two equations, we have,

$$\begin{aligned} e^{j\theta_0} x &= e^{j\theta_0} \sum_{i=1}^n \alpha_i x_i, \\ Ax + Bu &= A \sum_{i=1}^n \alpha_i x_i + B \sum_{i=1}^n \alpha_i u_i \\ &= \sum_{i=1}^n \alpha_i e^{j\theta_i} x_i, \end{aligned}$$

which implies,

$$\sum_{i=1}^n \alpha_i e^{j\theta_0} x_i = \sum_{i=1}^n \alpha_i e^{j\theta_i} x_i.$$

From this, we can conclude that $\alpha_i = 0$, for all $i$, because $\{x_i\}_{i=1}^n$ are linearly independent, and $\theta_i \neq \theta_0$ for all $i$. Therefore, $S_1 \cap S_2 = \{0\}$ which implies $\{v_i\}_{i=1}^{n+m}$ are linearly independent. Henceforth, $V = \sum_{i=1}^{n+m} v_i v_i^*$ satisfies the strict inequalities in (3), including $V \succ 0$, which implies strong duality holds from the Slater's constraint qualification [BV04]. □

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