RMM 21 Feb 08

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CDS 140b: Bifurcation Control w/ Mag & Rate Limits

Motivating Example: Moore-Greitzer model

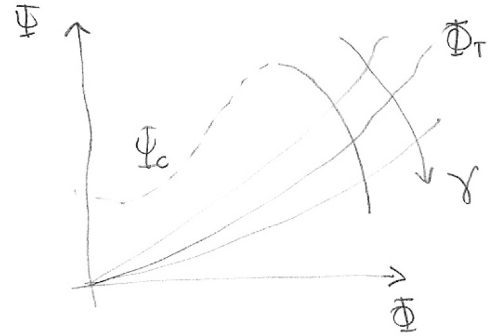
$$\dot{\Phi} = \frac{1}{l_c}\left(-\Psi + \Psi_c(\Phi) + \frac{J^2}{4}\frac{\partial^2 \Psi_c}{\partial \Phi^2}\right)$$

$$\dot{\Psi} = \frac{1}{4B^2 l_c}\left(\Phi - \Phi_T(\Psi, \gamma)\right)$$

$$\dot{J} = \frac{2}{m+\mu}\left(\frac{\partial \Psi_c}{\partial \Phi} + \frac{J}{8}\frac{\partial^3 \Psi_c}{\partial \Phi^3}\right) J$$

$$\Psi(\Phi) = a_0 + a_1 \Phi - a_3 \Phi^3$$

$$\Phi_T(\Psi) = \gamma\sqrt{\Psi}$$



$a_0 > 0 \quad a_1 \approx \frac{3}{2} \quad a_3 \approx \frac{1}{2}$

Equilibrium points

$J = 0$

$\psi_e = \Psi_c(\phi) \qquad \psi = \Psi - \Psi_e$

$\phi_e = \Phi_T(\psi) \qquad \phi = \Phi - \Phi_e$

$J_e = 0$

$m_c := \frac{\partial \Psi_c}{\partial \Phi}(\Phi_e) \qquad k_B = \frac{1}{4B^2}$

$m_T := \frac{\partial \Phi_T}{\partial \Psi}(\Psi_e) > 0$

Linearization

$$\frac{d}{dt}\begin{bmatrix}\phi \\ \psi \\ J\end{bmatrix} = \begin{bmatrix}\frac{m_c}{l_c} & -\frac{1}{l_c} & 0 \\ \frac{k_B}{l_c} & \gamma_e \frac{m_T}{l_c} & 0 \\ 0 & 0 & \frac{2m_c}{m+\mu}\end{bmatrix}\begin{bmatrix}\phi \\ \psi \\ J\end{bmatrix} + \begin{bmatrix}0 \\ \Phi_e \\ 0\end{bmatrix} u$$

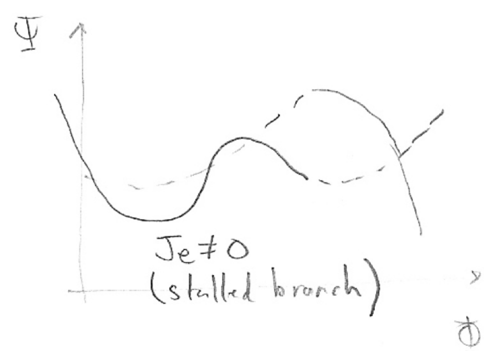
$J \neq 0$

$$J_e = \Psi_c' / \Psi_c''' = \frac{a_1 - 3a_3\Phi_e^2}{6a_3}$$

$$\Phi_e = \Phi_T(\Psi_e) = \gamma\sqrt{\Psi_e}$$

$$\Psi_e = \Psi_c(\Phi_e) + \frac{J_e^2}{4}\Psi_c''$$

$$= a_0 + a_1\Phi_e - (6a_3\Phi_e)\frac{J_e^2}{4} - a_3$$



Remarks

1. Linearization about $J=0$ is not reachable and if $m_c > 0$ there is an unstable, unreachable mode $\Rightarrow$ need nonlinear techniques.

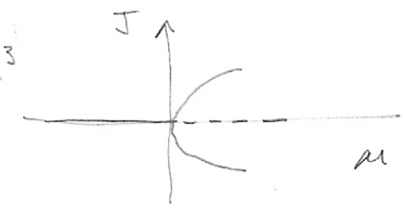
2. Can show that (Φ, Ψ) dynamics are stable if $m_c < \frac{1}{4B^2 m_T}$. Beyond this limit we get a Hopf bifurcation. In this case the linearization is reachable $\Rightarrow$ we can stabilize by modulating γ (throttle setting). However, $J=0$ goes unstable first ...

Bifurcation analysis

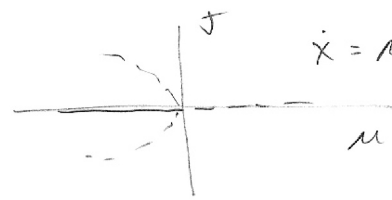
To understand what happens when the Hopf bifurcation has been stabilized, we need to sort out dynamics for $J \neq 0$.

To sort out sub- vs supercritical, look at the center manifold.

$\dot{x} = \mu x - x^3$



$\dot{x} = \mu x + x^3$



At $\mu = 0$, eq pt is stable

At $\mu = 0$, eq pt is unstable

$$\dot{J} = \alpha_1 (\gamma - \gamma_0) J + \alpha_2 J^2 + \text{h.o.t.}$$

$$\alpha_1 = \underbrace{\frac{2\sqrt{\Psi_e}\, \Psi_c''}{m + \mu}}_{<0} \qquad \alpha_2 = \frac{1}{4(m+\mu)} \left(\underbrace{\Psi_c'''}_{>0} + \underbrace{\frac{\gamma_0 (\Psi_c'')^2}{\sqrt{\Psi_e}}}_{>0} \right)$$

$\alpha_2 > 0 \Rightarrow$ unstable (subcritical)

Note that if we choose $\gamma = \gamma_0 + kJ$, then dynamics become

$$\dot{J} = \alpha_1 (\cancel{\gamma} - \gamma_0) J + (\alpha_2 + k\alpha_1) J^2 + \text{h.o.t.}$$

and if $\alpha_2 + k\alpha_1 < 0 \Rightarrow k > \left|\frac{\alpha_2}{\alpha_1}\right|$ stabilizes dynamics on center mfd $\Rightarrow$ supercritical bifurcation.

Remarks

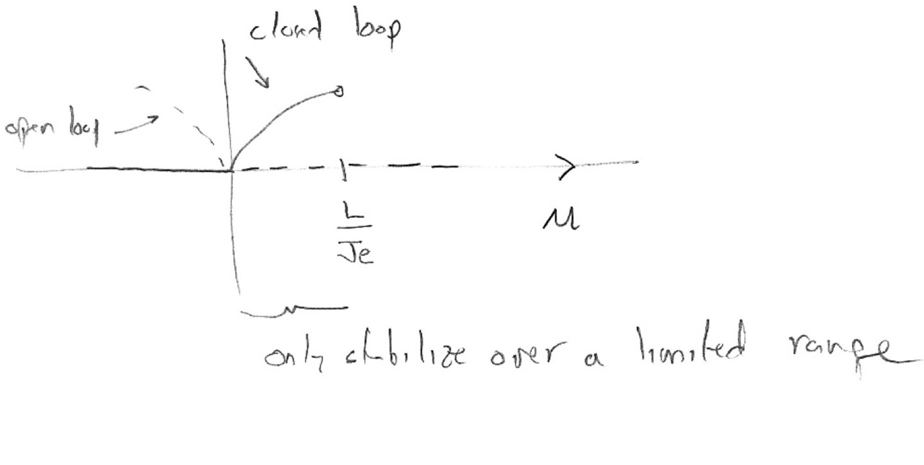
1. Since we have a J^2 term instead of J^3, actually get a transcritical bifurcation in J

2. Physically $J = A^2$, where A is the amplitude of the first mode of rotating stall

$$\dot{J} = 2A\dot{A} \Rightarrow \dot{A} = \frac{1}{2A}\left(\alpha_1(\gamma - \gamma_0) A^2 + (\alpha_2 + k\alpha_1) A^4 + \dots\right)$$
$$= \alpha_1(\gamma - \gamma_0) A + (\alpha_2 + k\alpha_1) A^3 + \dots$$

so pitchfork bifurcation in A.

3. For $\gamma < \gamma^*$, get $J_e > 0 \Rightarrow u_e = kJ_e$. If $|u_e| < L$, can only stabilize up to some point



Bifurcation control

Consider a nonlinear control system

$$\dot{x} = f(x, u, \mu) \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R},\ \mu \in \mathbb{R}$$

Assumptions:

A1. $f(x, u, \mu)$ is C^4 in (x, u) and C^2 in μ

A2. For $u = 0$, $\exists$ nominal equilibrium solution $x_e(\mu)$

A3. $\exists$ eigenvalue $\lambda(\mu)$ for the linearization at $(x_e(\mu), 0)$ with $\lambda(0) = 0$, $\frac{d\lambda}{d\mu}(0) \neq 0$

A4. The eigenspace for $\lambda(\mu)$ is linearly unreachable and all other eigenspaces are linear reachable

Prop Under assumptions A1-A4, $\exists$ a linear change of coordinates $(y, z) = Tx$ such that the dynamics have the form

$$\begin{aligned}\dot{y} = \mu y &+ q_{11} y^2 + (q_{12} z) y + q_{13} y u \\ &+ z^T q_{22} z + (q_{23} z) u + q_{33} u^2 + \\ &\text{third order terms } (q_{ijk}) + \text{higher order terms}\end{aligned}$$

$$\begin{aligned}\dot{z} = Az + Bu &+ r_{11} y^2 + (r_{12} z) y + r_{13} y u + \\ &\text{remaining second order terms} + \text{higher order terms}\end{aligned}$$

$$u = k_1 y + k_2 z + k_{11} x^2 + (k_{12} z) x + z^T k_{22} z + \cdots$$

Pf Bookkeeping

Approach: Given overall form,

1. Divide into classes based on coefficients & conditions
2. For each class, compute dynamics on center manifold ($\mu = 0$)
3. Convert to appropriate normal form and analyze
4. Compute feedback coefficients to stabilize

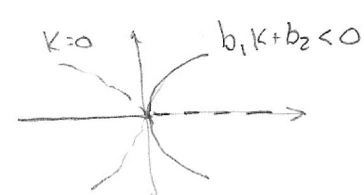
Example:

$$\Upsilon_1 = q_{12}^n - q_{13} \det(A) \leftarrow \text{quadratic term on center mfd}$$

$$\alpha_0 = q_{111} - \left([0 \; q_{12}^1 \cdots q_{12}^{n-1}] + q_{13} [1 \; a_1 \cdots a_{n-1}] \right) \cdot r_{11} \leftarrow \text{cubic term}$$

SS-1: $q_{11} = 0, \; \alpha_0 < 0 \Rightarrow \dot{y} = \mu y + b_1 x u + b_2 x^3, \; b_2 < 0$

Already supercritical. (w/ $u = 0$)



SS-2: $q_{11} = 0, \; \Upsilon_1 \neq 0, \; \alpha_0 > 0$

$$\dot{y} = \mu + b_1 y u + b_2 y^3$$

$$u = K y^2$$

$$b_1 = \frac{\Upsilon_1}{\det(A)}$$

$$b_2 = q_{111} + \frac{1}{\det A}\left(q_{12}^n [1 \; a_1 \cdots a_{n-1}] - \det A [0 \; q_{12}^1 \cdots q_{12}^{n-1}] \right) r_{11}$$

Choose K such that $(b_1 K + b_2) < 0 \Rightarrow$ supercritical pitchfork

SS-3: $q_{11} = 0, \; \Upsilon_1 = 0, \; \Upsilon_2 = q_{22}^{nn} + q_{23}^n \det A + q_{33} (\det A)^2 = 0$

+ more conditions

$$\dot{y} = \mu y + \alpha_0 y^3 + \alpha_1 y^2 u + \alpha_2 y u^2 + \alpha_3 y^3$$

$$u = K y$$

Etc

Thm (Abed & Fu, 1987) Given a nonlinear control system

$$\dot{x} = f(x, u, \mu) = L_0 x + \mu L_1 x + u \tilde{L}_1 x + Bu + Q_0(x,x) + \mu^2 L_2 x + \mu Q_1(x,x) + u \tilde{Q}_1(x,x) + C_0(x,x,x) + \ldots$$

let l and r be left and right eigenvectors of L_0. Assume A1 - A3 hold and $lB \neq 0$. Then $\exists$ smooth feedback $u = \alpha(x)$ containing only quadratic & cubic terms such that the resulting bifurcation is supercritical.

Only cover if there is time available

Remarks

1. Yet another special case of basic approach
2. For this thm, turns out $lB \neq 0 \Rightarrow$ linearization is controllable $\Rightarrow$ could just stabilize w/ linear feedback

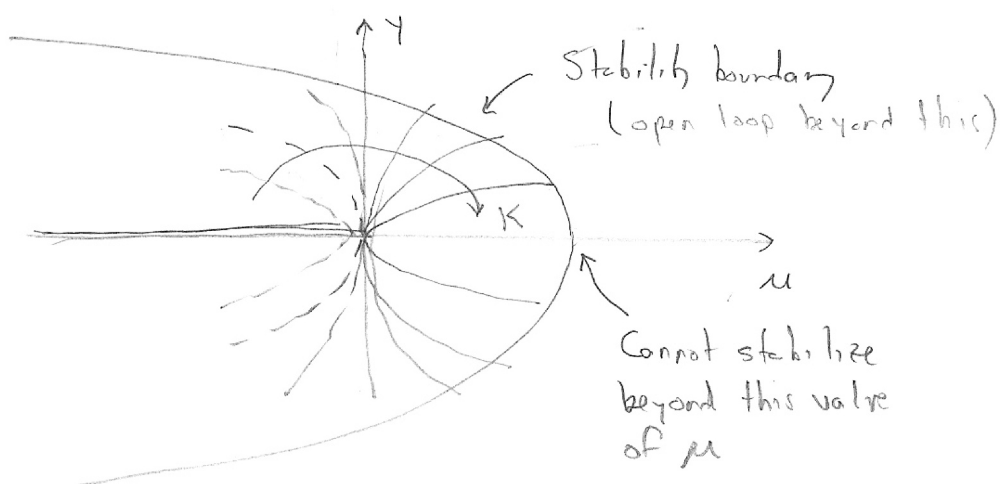
Magnitude Limits

$$\dot{x} = f(x, u, \mu) \qquad |u| < L$$

Local form of equations remains unchanged

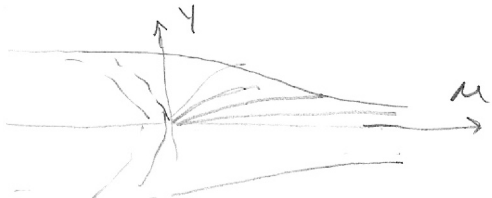
SS-2: $\dot{y} = \mu + b_1 y u + b_2 y^3$

$$u = \begin{cases} L & \text{if } Ky^2 \geq L \\ Ky^2 & \text{if } |Ky^2| < L \\ -L & \text{if } Ky^2 \leq -L \end{cases}$$



Remarks

1. Stability boundary determined by specific form of dynamics



SS- w/ limits

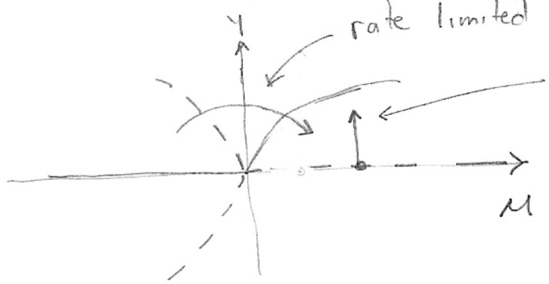
SS-3 case can stabilize to arbitrarily large μ

2. Remember that dynamics are taking place on center mfd and we ignored higher order terms $\Rightarrow$ approximate results only.

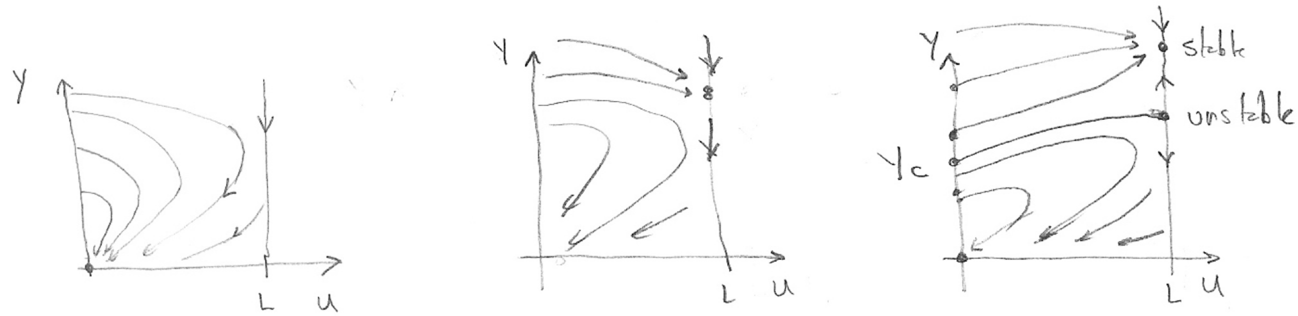
Rate limits

$$\dot{x} = f(x, u, \mu) \qquad |u| < L \qquad |\dot{u}| < M$$

Intuition: need to change bifurcation quickly enough to "beat" instability



Better approach: Look at phase portrait in terms of y & u



$\mu < 0$ — Single stable eq pt

$\mu = 0$

$\mu > 0$ — If y(0) starts large, can't change u quickly enough to avoid mag limit

Remarks:

1. Can compute stability boundary using polynomial approximations
2.