CDS110a Midterm Review

Sawyer Fuller
11/2/07

I. overview

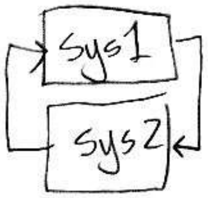
1. Central control systems ideas:
feedback, input-output behavior of systems and dynamics

a) dynamics: change over time

b) input-output: $u \rightarrow \boxed{\text{system}} \rightarrow y$
(input) (output)

example of a dynamic input-output system:
a linear ODE $y'' + 2y' + y = u + u'$
(can write output explicitly: $y = -y'' - 2y' + u + u'$)

c) feedback



* no cause & effect,
⇒ must consider system as a whole

benefits of feedback:
- resilience to external disturbance
- resilience to internal variation
  * biggie result: can get linear behavior from variable, nonlinear components!

2. A control system:
bases correcting actions on the difference between the actual and desired performance.

3. the course so far has mostly been concerned so far with dynamic behavior, with a little bit of feedback/IO
- equilibrium points
- stability of equilibrium pts
- a little bit of state feedback

II. Modeling and dynamics.

1. usually I/O blocks have hidden state that is not directly measurable.

    example: another ODE: $x'' + x = u$ $\quad u \to \boxed{sys} \to y$

    $y = 2x' + x$

    (don't know x from output)

2. state space form
    - state is set of vars that characterize motion of system
    - state space is set of all possible states
    - form:

    $$\dot{x} = f(x,u) \qquad x \in \mathbb{R}^n \text{ (state)}$$
    $$y = h(x,u) \qquad y \in \mathbb{R}^m \text{ (output)}$$
    $$u \in \mathbb{R}^p \text{ (input)}$$

    (full, nonlinear case)

    - example: ODE $\to$ S.S.

    $$\ddot{z} = z - z^2$$

    set $x_1 = z$, $x_2 = \dot{z}$ $\Rightarrow$ $\dfrac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 - x_1^2 \end{bmatrix}$

III. Extended example

- equilibrium points
- stability

Problem:

Consider the system: (similar to Excercise 3.8 in book)

$$(1)\quad \dot{x}_1 = a(x_1 - x_2)$$
$$(2)\quad \dot{x}_2 = b(x_2 - x_1) - c\frac{x_2}{d+x_2} + e + u$$
$$\Big\} = F(x,u)$$

$a = .0312$
$b = 2.174$
$c = 26.304$
$d = 0.1$
$e = 23.913$

where $a, b, c, d, e > 0$, $c = b\frac{(d+1)^2}{d}$

Find an equilibrium point, analyze stability, and if unstable, use state feedback to stabilize.

Solution:

equilibrium points at $\dot{x}_1 = \dot{x}_2 = 0$ (assuming $u=0$)

$$(1)\quad \dot{x}_1 = 0 = a(x_1 - x_2) \Rightarrow x_1 = x_2$$
$$(2)\quad \dot{x}_2 = 0 = -c\frac{x_2}{d+x_2} + e$$

$$\frac{x_2}{d+x_2} = \frac{e}{c} \Rightarrow x_2 - \frac{e}{c}x_2 = \frac{ed}{c} \Rightarrow x_2 = \frac{ed/c}{1-\frac{e}{c}} = 1$$
$$\Rightarrow x_1 = 1$$

only eq. point $x_e = (1,1)$

analysis of stability at $(1,1) = x_e$

- linearize to find behavior of linear system near x_e.

Linearization

taylor series: $f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \ldots$

n-d: $\vec{x}$ a vector: $f(\vec{x}) \approx f(\vec{a}) + \frac{\partial F}{\partial x}(\vec{a})(\vec{x}-\vec{a}) + \ldots$

where $\frac{\partial F}{\partial x} = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix}$

the two terms $f(x) \approx f(a) + \frac{\partial F}{\partial x}(a)(x-a)$

are the linearization of the dynamics.

in controls, we often want a system to get somewhere and <u>stop</u>.

this only happens, at equilibrium points

$\dot{x} = f(x)$ where $\dot{x} = 0 \Rightarrow$ where $f(x_e) = 0$.

you can push a system to a desired equilibrium point with proper baseline input, but for now we're going to assume we're already at the desired equilib. point, we just want to make sure it's stable.

the linearization becomes:

$$f(x) \approx \cancel{f(x_e)}^{\,0} + \frac{\partial F}{\partial x}(x_e)(x - x_e)$$

or $\dot{x} \approx A(x - x_e)$ where $A = \left.\frac{\partial f}{\partial x}\right|_{x_e, u_e}$

setting $z = x - x_e$,
differentiating, $\dot{z} = \dot{x}$ $\Big\}$ $\dot{z} \approx Az$

similarly, $B = \left.\frac{\partial f}{\partial u}\right|_{x_e, u_e}$, $C = \left.\frac{\partial h}{\partial x}\right|_{x_e, u_e}$, $D = \left.\frac{\partial h}{\partial u}\right|_{x_e, u_e}$

Thus, full linearized system is

(3) $\begin{aligned} \dot{z} &= Az + Bv \\ w &= Cz + Dv \end{aligned}$ $\quad \begin{aligned} w &= y - y_e \\ v &= u - u_e \end{aligned}$

Remarks:

1. definition of linear input-output behavior:

if: $x \rightarrow y$
then $ax \rightarrow ay$
if $x_1 \rightarrow y_1$
and $x_2 \rightarrow y_2$
then $x_1 + x_2 \rightarrow y_1 + y_2$

example linear function: $y = \alpha x$
- the above conditions are satisfied.

example nonlinear function:
$y = x^2 : x \rightarrow x^2$
then $ax \rightarrow a^2x^2 \neq ax^2$
$x_1 + x_2 \rightarrow x_1^2 + 2x_1x_2 + x_2^2 \neq x_1^2 + x_2^2$

2. solution to linear system (3) is <u>convolution equation</u>

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)$$

if no input ($u=0$), solution is just $y(t) = Ce^{At}x_0$.

3. stability of system with no input depends on the eigenvalues of the A matrix.

if A is diagonalizeable with transform T, $AT = T\Delta$, $\Delta = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$
$A = T\Delta T^{-1}$

then solution $y = Ce^{At}x_0$

$$= C\left(I + At + A^2t^2 + \dots\right)x_0$$
$$= C\left(TT^{-1} + T\Delta T^{-1}t + (T\Delta \cancel{T^{-1}})(\cancel{T}\Delta T^{-1})t^2 + \dots\right)x_0$$
$$= CT\left(I + \Delta t + \Delta^2t^2 + \dots\right)T^{-1}x_0 = CTe^{\Delta t}T^{-1}x_0$$
$$= CT\begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix}T^{-1}x_0$$

goes to zero if and only if all $Re\{\lambda_i\} < 0$.

- can give similar proof using Jordan form of A if A is not diagonalizeable.

so, back to example, the system is stable if the eigen values of A are in the left half of the complex plane.

~~first, let~~ $\cancel{z = x - x_e = \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}, \quad \dot{z} = \dot{x}}$

find $A = \frac{\partial f}{\partial x} = \begin{bmatrix} a & -a \\ -b & b - c\,\frac{(d+x_2)-x_2}{(d+x_2)^2} \end{bmatrix} = \begin{bmatrix} a & -a \\ -b & b - c\,\frac{d}{(d+x_2)^2} \end{bmatrix} \Bigg|_{x_e = (1,1)}$

$$= \begin{pmatrix} a & -a \\ -b & \underbrace{b - c\,\frac{d}{(d+1)^2}}_{=b} \end{pmatrix} = \begin{bmatrix} a & -a \\ -b & 0 \end{bmatrix}$$

and we know $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ by inspection.

eigvals of A: $\det(A - sI) = 0 \Rightarrow (a-s)(-s) - (-a)(-b) = 0$

$$s^2 - as - ab = 0$$

$$s = \frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 + 4ab}$$

unstable for all $a, b > 0 \Rightarrow$ unstable.

let's consider state feedback to stabilize.

is it reachable? Reachability test: reachability matrix $[B \ AB]$ full rank. can test with $\det[B \ AB]$ if not zero $\Rightarrow$ full rank.

(matrix will be square for single input systems)

$$[B \ AB] = \begin{bmatrix} 0 & -a \\ 1 & 0 \end{bmatrix}$$

test: $\det\begin{bmatrix} 0 & -a \\ 1 & 0 \end{bmatrix} = 0 + a \neq 0$

$\Rightarrow$ reachable.

to stabilize, since system is reachable, we can place eigenvalues to be any value we want. if the eigenvalues of the linearized system are stable, then the full nonlinear system is stable (in the neighborhood of that equilibrium point!)

consider state feedback of form $u = -Kx = -k_1x_1 - k_2x_2$

$$K = [k_1 \ \ k_2]$$

use $z = x - x_e$, then linearized system is

$$\begin{aligned} \dot{z} &= Az + Bu \\ &= Az - BKz \\ &= (A - BK)z \end{aligned}$$

suppose we want our eigenvalues s to satisfy the characteristic equation $s^2 + 2\zeta w_0 s + w_0^2 = 0$ ($\Rightarrow$ stable eigenvalues for $w_0, \zeta > 0$)

find matrix K:

want $\det((A-BK)-sI) = 0$ to have same form.

$$A - BK = \begin{bmatrix} a & -a \\ -b & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$

$$= \begin{bmatrix} a & -a \\ -b & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ K_1 & K_2 \end{bmatrix} = \begin{bmatrix} a & -a \\ -b-K_1 & -K_2 \end{bmatrix}$$

$$\det(A-BK-sI) = \det\begin{bmatrix} a-s & -a \\ -b-K_1 & -K_2-s \end{bmatrix} = 0$$

$$= (a-s)(-K_2-s) - (-a)(-b-K_1)$$
$$= s^2 - as + K_2 s - aK_2 - ab - aK_1$$
$$= s^2 + (-a+K_2)s + (-aK_2 - ab - aK_1) = 0$$

(1) $-a + K_2 = 2\zeta w_0$

(2) $-aK_2 - ab - aK_1 = w_0^2$

solve: (1) $K_2 = +2\zeta w_0 + a$

(2) $-2a\zeta w_0 + a^2 - ab - aK_1 = w_0^2$

$-aK_1 = w_0^2 + ab + a^2 + 2a\zeta w_0$

$K_1 = -\frac{w_0^2}{a} - b - a - 2\zeta w_0$

choose $w_0 = 1$

$\zeta = 0.5$

so $u = K_1 z_1 - K_2 z_2$

$= -K_1(x_1 - 1) - K_2(x_2 - 2)$

also, we could have used matlab function: $K = \text{place}(A, B, \underbrace{[s_1\ s_2 \cdots s_n]}_{\text{desired eigenvals.}})$

show stability with a Lyapunov function at x_e.

propose: $V = z^T P z$ V is scalar, to prove stability need $V > 0$ for $z \neq 0$, $V = 0$ at $z = 0$. (: pos def) and $\dot{V} \leq 0$ for $z \neq 0$ (neg def.)

then $\dot{V} = \frac{dV}{dt} = \dot{z}^T P z + z^T P \dot{z}$

let $\hat{A}$ be matrix with state feedback $\hat{A} = A - BK$

and nonlinear system is $\dot{z} = \hat{A} z + \tilde{f}(z)$ where $\tilde{f}$ is higher-order terms

$$\tilde{f}(z) = f(z + x_e) - Az$$

then $\dot{V} = (\hat{A}z + \tilde{f}(z))^T Pz + z^T P(\hat{A}z + \tilde{f}(z))z$

$= z^T(\hat{A}^T P + P\hat{A})z$ + higher-order $O(3)$

$= -z^T Q z$

need only $Q > 0$ for $\dot{V} < 0$

choose $Q = I \Rightarrow$ solve for P.

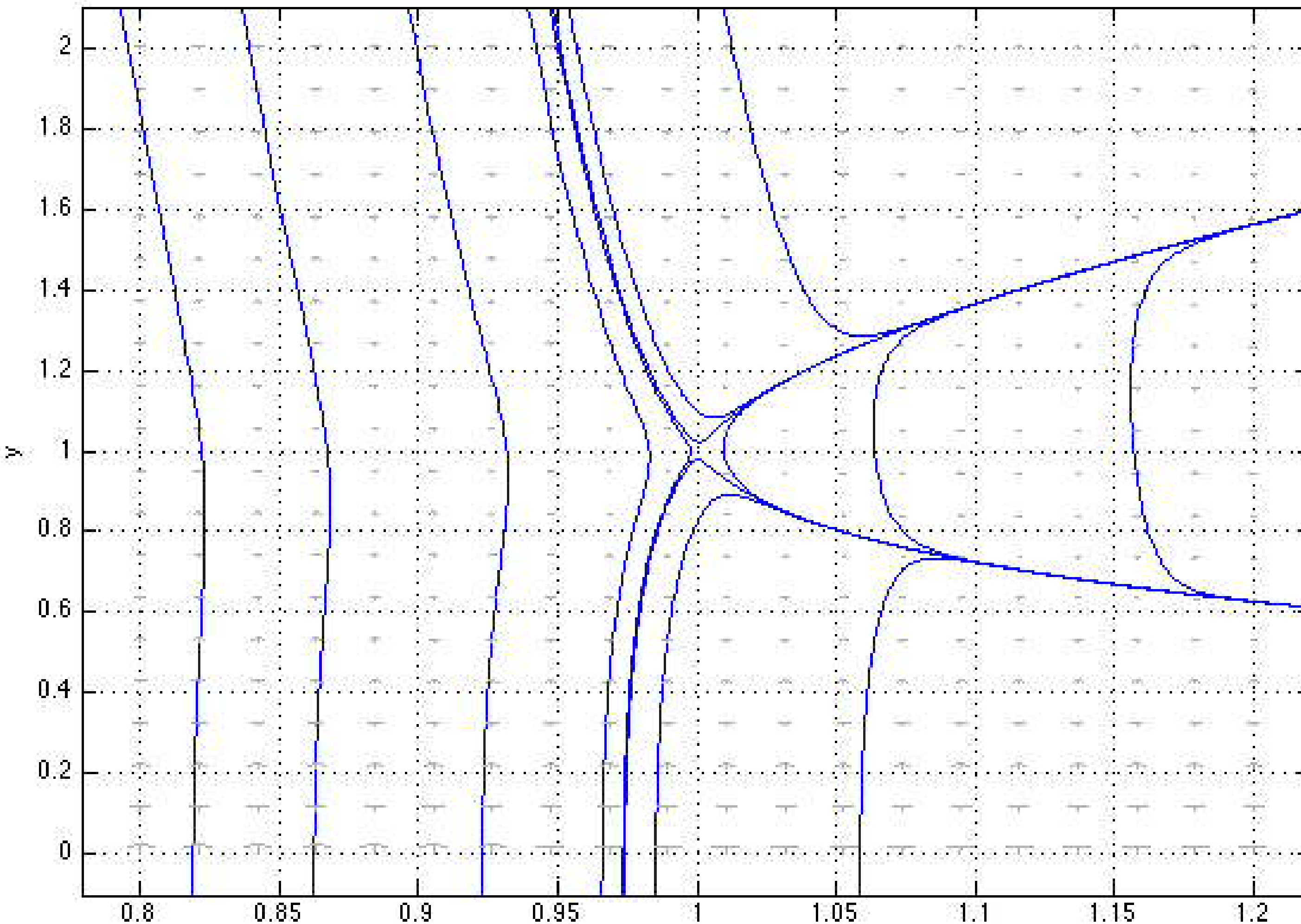
$\Rightarrow$ will find one if $\hat{A}$ stable (which it is here)

$P = \text{lyap}(A, Q)$ in MATLAB

proves stability of nonlinear system because $O(3)$ terms will always be smaller than $O(2)$ terms in any sufficiently small neighborhood.

$x' = a\,(x - y)$
$y' = b\,(y - x) - b\,(d + 1)^2/d\; y/(d + y) + e$

$a = 1.5/48 \qquad b = 1.5/.69$
$e = 16.5/.69 \qquad d = .1$

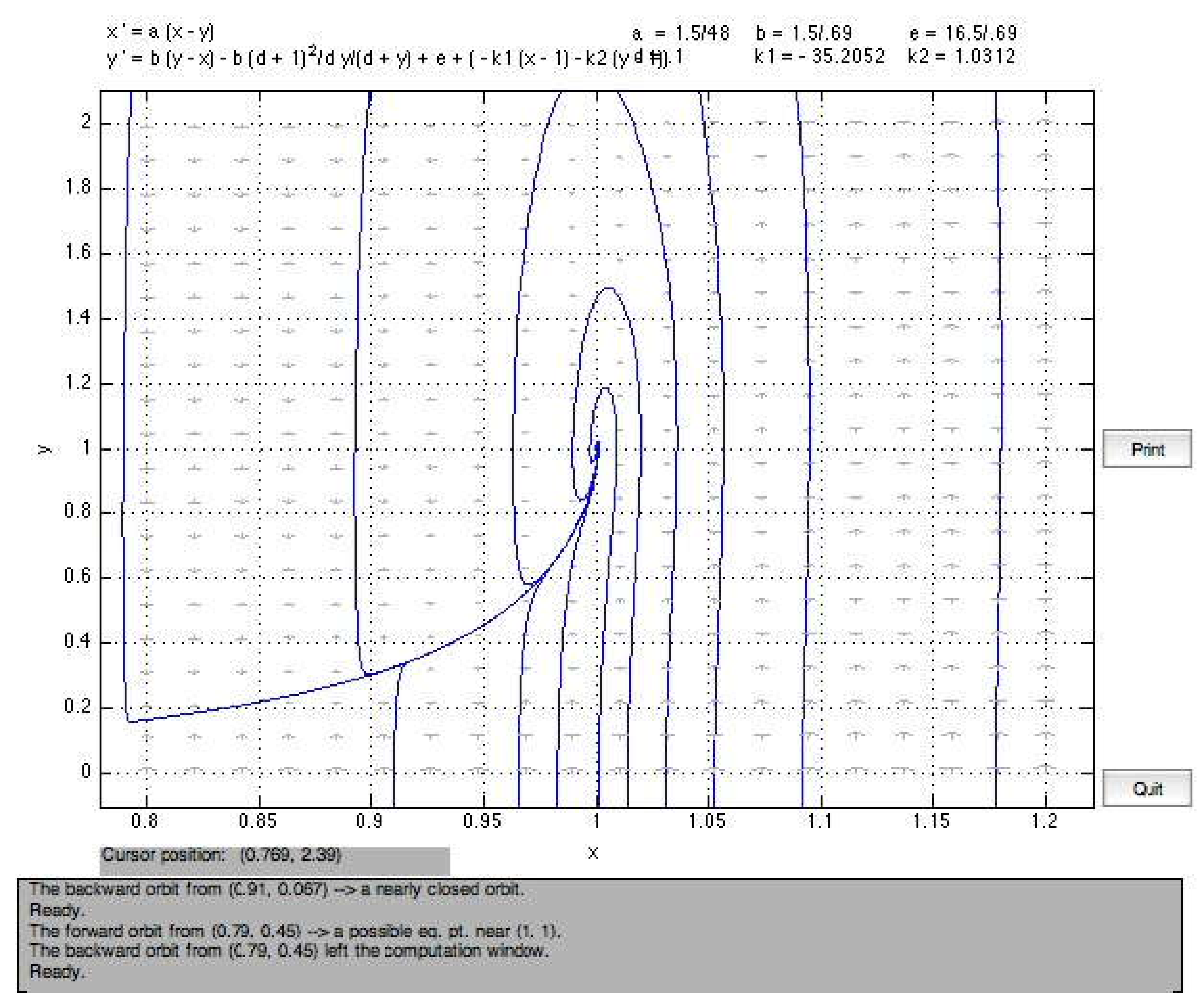


Print

Quit

Cursor position: (0.819, 2.35)

The backward orbit from (0.82, 0.8) --> a nearly closed orbit.
Ready.
The forward orbit from (1.1, 1) left the computation window.
The backward orbit from (1.1, 1) left the computation window.
Ready.


x ' = a (x - y)
y ' = b (y - x) - b (d + 1)^2/d y/(d + y) + e + (- k1 (x - 1) - k2 (y - 1)).1
a = 1.5/48
b = 1.5/.69
e = 16.5/.69
k1 = - 35.2052
k2 = 1.0312
Print
Quit
y
x
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15
1.2
Cursor position: (0.769, 2.39)
The backward orbit from (0.91, 0.067) --> a nearly closed orbit.
Ready.
The forward orbit from (0.79, 0.45) --> a possible eq. pt. near (1, 1).
The backward orbit from (0.79, 0.45) left the computation window.
Ready.