RMM 28 Feb 08

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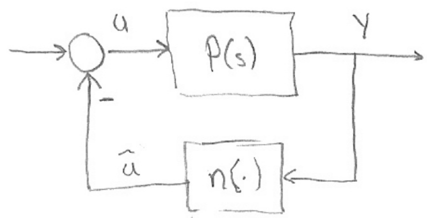
CDS 140b: Describing Functions

Control of limit cycles



Questions:

1. When is there a limit cycling solution?
2. When is the limit cycle stable?
3. How can we control the size of the limit cycle?

(Single-input) Describing function method

Suppose $P(s)$ stable and $n(y)$ is a static (single-valued) function

Assume $\tilde{y}(t) = A\sin\omega t$, expand $\tilde{y}$ as Fourier series

$$\tilde{u}\left(t + \frac{2\pi}{\omega}\right) = n\left(y\left(t+\frac{2\pi}{\omega}\right)\right) = n(y(t)) = \tilde{u}(t)$$

$$\tilde{u}(t) = \sum_{k=0}^{\infty} a_k \sin(k\omega t + \phi_k) = \sum_{k=0}^{\infty} \alpha_k \sin(k\omega t) + \beta_k \cos(k\omega t)$$

Assume, for simplicity, that $n(\cdot)$ is an odd function $\Rightarrow$ $n(y) = -n(-y)$

$\Rightarrow n\left(y\left(t+\frac{\pi}{\omega}\right)\right) = -n(y(t)) \Rightarrow \beta_j = 0$ and also $\alpha_0 = 0$

$$\tilde{u}(t) = a_1 \sin(\omega t) + \underbrace{\cancel{a_2}}_{0}\sin(2\omega t) + a_3 \sin(3\omega t) + \cdots$$

$$= N(A)\cdot A\sin(\omega t) + \text{harmonics}$$

$$N(A) = \frac{a_1}{A} = \frac{1}{A}\int_0^{2\pi} n(A\sin\theta)\sin\theta \, d\theta \leftarrow \underline{\text{Describing fcn for } n(\cdot)}$$

Now run through linear system

assume small

$$y = -|H(i\omega)|\, N(A) A \sin(\omega t + \arg H(i\omega)) - \cancel{|H(i3\omega)|}\,(3^{rd}\ \text{harmonic}) \cdots$$

Limit cycle condition: $y = \tilde{y}$

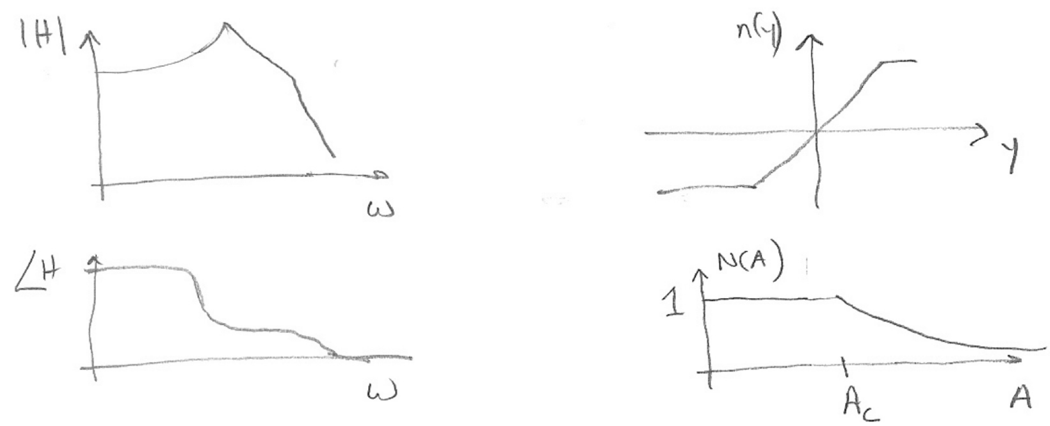
$$-|H(i\omega)|\,N(A)\,A \sin(\omega t + \arg H(i\omega)) = A \sin \omega t$$

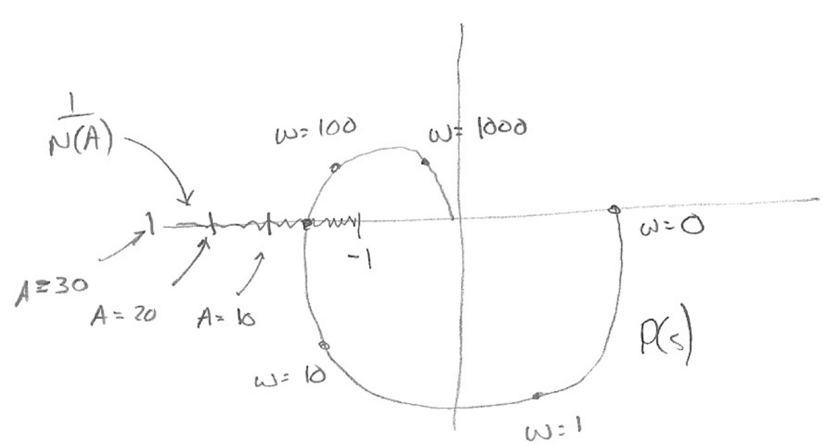
$$\Rightarrow \quad |H(i\omega)|\,N(A) = 1 \qquad \arg H(i\omega) = \pi$$

$$\boxed{H(i\omega) = -\frac{1}{N(A)}}$$

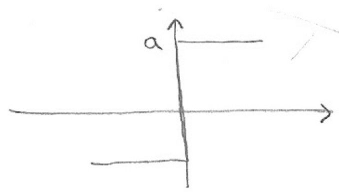
Remarks

1. Assumes third (& higher) harmonics are small
2. Get magnitude and frequency of oscillation by solving algebraic eqs
3. Graphical method using Nyquist plot





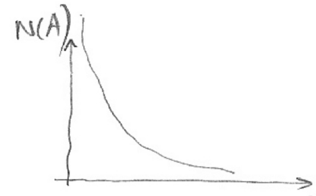
Describing function examples

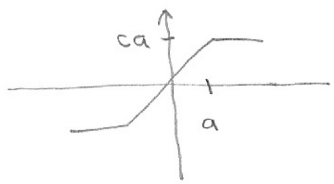


$u(t) = A \sin \omega t$

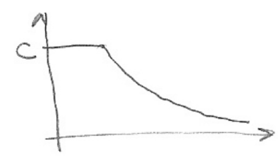
$y(t) = \mathrm{sqwv}(\omega t)$

$\Rightarrow a_1 = \frac{1}{2\pi}\int_0^{2\pi} a \sin^2\theta \, d\theta = \frac{4a}{\pi} \Rightarrow N(A) = \frac{4a}{\pi A}$

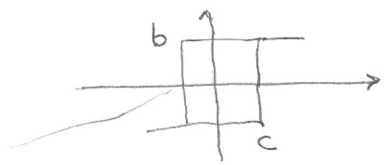




$N(A) = \begin{cases} c & \alpha = \frac{A}{a} < 1 \\ c k(\alpha) & \alpha \geq 1 \end{cases}$



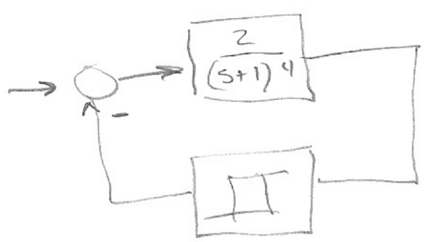
$k(\alpha) = \frac{2}{\pi} \sin^{-1}\left(\frac{1}{\alpha}\right) + \frac{1}{\alpha}\sqrt{1 - \left(\frac{1}{\alpha}\right)^2}$

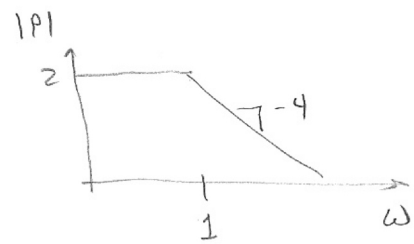


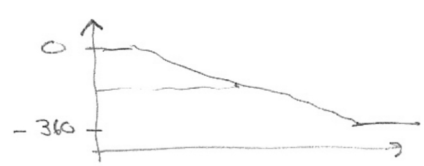
$N(A) = \frac{4b}{A\pi}\left(\sqrt{1 - \frac{c^2}{A^2}} - i \frac{c}{A}\right)$

imaginary part! (ok)

Example: Linear system with relay feedback

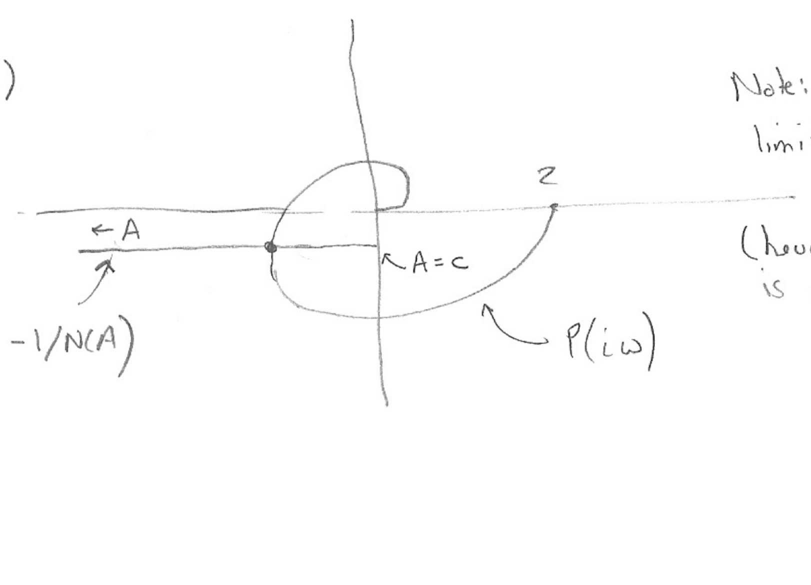






$\frac{1}{N(A)} = \frac{\pi\sqrt{A^2 - c^2}}{4b} + i \frac{\pi c}{4b}$

$P(j\omega) = -\frac{1}{N(A)}$



Note: always get limit cycle at $A \geq c$ and (heuristically) it is always stable.

Stability of limit cycles via describing functions

Q: How do we check if limit cycle is <u>stable</u>?

A1: Poincare map and/or linearization around solution

$$\dot{x} = f(x)$$

$$\dot{x}_d = f(x_d) \quad \leftarrow \text{limit cycle}$$

$$\dot{e} = f(x) - f(x_d) = F(e + x_d(t)) \approx A(t)e$$

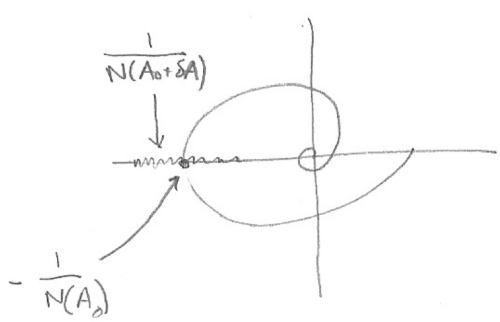
For periodic system, can use Floquet theory:

Let $\Phi(t,0)$ be the (T-periodic) fundamental matrix for $\dot{e} = A(t)e$

Can show that $\Phi(t,0) = P(t)e^{Ft}$ where $P(t) = P(t+T)$

Check that $\lambda(e^{TF})$ have magnitude less than one

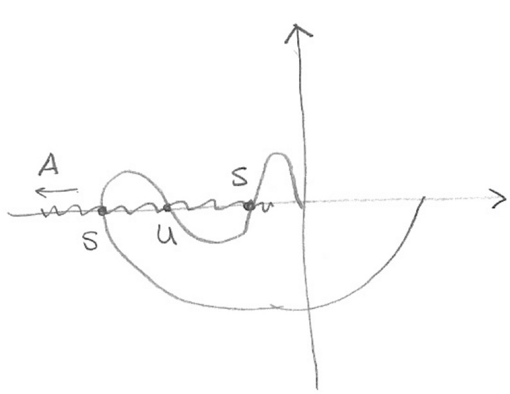
A2: Graphical check



Suppose we perturb A to $A_0 + \delta A$

$\Rightarrow \frac{1}{N(A_0)}$ becomes $\frac{1}{N(A_0+\delta A)}$

- If $\frac{1}{N(A_0+\delta A)}$ "stabilizes" loop (removes encirclement) $\Rightarrow$ stable limit cycle
- If $\frac{1}{N(A_0+\delta A)}$ gives unstable encirclement $\Rightarrow$ unstable limit cycle



Describing function theory (Mees)

Suppose (ω_0, A_0) satisfies $P(i\omega_0) = -1/N(A_0)$ and that the intersection of the two curves is transverse. Define

$$\rho(\omega)^2 = \sum_{k=3,5,9,\ldots} |P(ik\omega_0)|^2 \qquad \text{"gain of harmonics"}$$

$$p(A)^2 = \|n(A\sin t)\|_2^2 - |A\,N(A)|^2 \qquad \text{"error in } 1^{st} \text{ harmonic"}$$

$$q(A,\varepsilon) = \|m(A\sin t, \varepsilon)\|_2 \qquad m(x,\varepsilon) = \max\{|n(x+\varepsilon) - n(x)|,\ |n(x-\varepsilon) - n(x)|\}$$

"slope bound"

Find ε such that for all (ω, A) near (ω_0, A_0)

$$\rho(\omega)\,(p(A) + q(A,\varepsilon)) \le \varepsilon$$

and let Ω be the set of (ω, A) such that

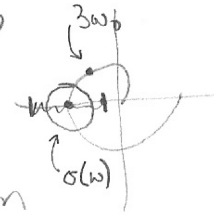
$$|N(A) + 1/G(i\omega)| \le q(A,\varepsilon)/A$$

<u>Thm</u> (Mees, 1970s) Suppose Ω is bounded and there exists a unique $(\omega_0, A_0) \in \Omega$ satisfying balance equation. Then there exists a periodic solution of the form $y(t) = A\sin(\omega t) + y^*(t)$ with remnant $\|y^*\|_\infty \le \varepsilon$.

<u>Pf</u> Reduce to contraction mapping theorem (which generates ρ, p, q)

<u>Remarks</u>

1. Roughly speaking need harmonics to die off sufficiently fast
2. Can also construct graphical version: "complete intersection"
3. Can also get a describing fcn version of Hopf bifurcation



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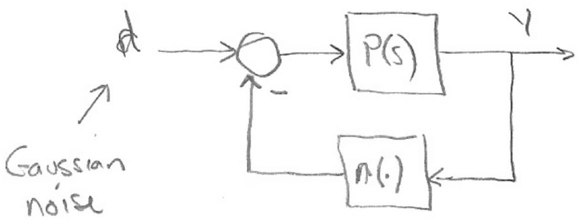
Random Input Describing Functions



Q0: how does noise propogate thru NL?

Q1: how do we distinguish lightly damped system from limit cycle?

Q2: Can noise "stabilize" a NL system?

Construct input that contains sinusoids plus noise:

$$y = B + A\sin(\omega t + \phi) + r(t)$$

B = bias, A = amplitude

$r(t)$ = stationary Gaussian random process with variance σ^2, corr $\rho(\tau)$

ϕ = uniform random variable

$$n(y(t)) = N_B B + N_A A\sin(\omega t + \phi) + N_R r(t)$$

N_B: bias gain; N_A: sinusoidal gain; N_R: stochastic gain — "Random input DFs"

Example of a multi-input describing function method.

$$N_B(B,A,\sigma) \equiv \frac{1}{B} E\{f(y)\} = \frac{1}{(2\pi)^{\frac{3}{2}}\sigma B}\int_0^{2\pi}\int_{-\infty}^{\infty} f(B + A\sin\theta + r)\, e^{-\frac{r^2}{2\sigma^2}}\, dr\, d\theta$$

$$N_A(B,A,\sigma) = \frac{2}{A} E\{f(y)\sin\theta\}$$
$$= \frac{2}{(2\pi)^{3/2}\sigma A}\int_0^{2\pi}\int_{-\infty}^{\infty} f(B + A\sin\theta + r)\sin\theta\, e^{-\frac{r^2}{2\sigma^2}}\, dr\, d\theta$$

$$N_R(B,A,\sigma) = \frac{1}{\sigma^2} E\{f(y)\, r\} = \cdots$$

Remarks

1. Can show that this recovers standard describing function if we take $\sigma = 0$
2. In general, very difficult to compute N_B, N_A, N_R in closed form

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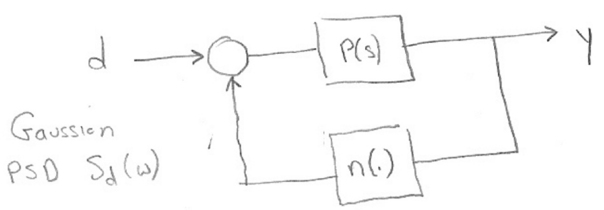
Random Signal Case

Consider the case when $A = 0 \Rightarrow$ how does noise propogate?



Linear case:

$$H_{yd}(s) = \frac{P}{1 + PN}$$

$$S_y(\omega) = H_{yd}(-i\omega)\, S_d(\omega)\, H_{yd}(i\omega)$$

$$\sigma_y = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_y(\omega)\, d\omega$$

Nonlinear case: replace N with $N_R(\sigma_y)$

$$\sigma_y = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_{yd}(-i\omega)\, S_d(\omega)\, H_{yd}(i\omega) \qquad H_{yd}(s) = \frac{P(s)}{1 + P(s)\, N_R(\sigma_y)}$$

Get an algebraic equation that we can solve for σ_y, then plug into $N_R(\sigma_y)$ and compute $S_y(\omega)$ = output spectral density fcn.

Note: If $H_{yd}(s)$ is unstable, will get $\sigma_y \to \infty$

Random Signal + Sinusoid

Suppose we have both limit cycle & noise.

$$y(t) = A \sin(\omega t + \phi) + r(t)$$

Look for solutions of coupled equations

$$\sigma_y = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_{yd}(-i\omega)\, S_d(\omega)\, H_{yd}(\omega) \qquad H_{yd}(s) = \frac{P(s)}{1 + P(s)\, N_R(A, \sigma_y)}$$

$$N_A(A, \sigma_y)\, P(j\omega_0) = -1$$

If we can find A, σ_y and ω_0 that satisfy all equations then get description of $y(t)$

Remarks:

1. Some times happens that $S_d(\omega)$ cause unstable (noiseless) system to be stable (!). Stability by dithering
2. Similarly, can get system w/ $N_R(0, \sigma_y)$ destabilizing to be stable.