Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. **Perko, Section 3.4, problem 1**: Show that $\gamma(t) = (2 \cos 2t, \sin 2t)$ is a periodic solution of the system
   \[
   \begin{align*}
   \dot{x} &= -4y + x \left(1 - \frac{x^2}{4} - y^2\right) \\
   \dot{y} &= x + y \left(1 - \frac{x^2}{4} - y^2\right)
   \end{align*}
   \]
   that lies on the ellipse $(x/2)^2 + y^2 = 1$ (i.e., $\gamma(t)$ represents a cycle $\Gamma$ of this system). Then use the corollary to Theorem 2 in Section 3.4 to show that $\Gamma$ is a stable limit cycle.

2. **Perko, Section 3.4, problem 3a**: Solve the linear system
   \[
   \dot{x} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} x
   \]
   and show that at any point $(x_0, 0)$ on the $x$-axis, the Poincare map for the focus at the origin is given by $P(x_0) = x_0 \exp(2\pi a / |b|)$. For $d(x) = P(x) - x$, compute $d'(0)$ and show that $d(-x) = -d(x)$.

3. **Perko, Section 3.5, problem 1**: Show that the nonlinear system
   \[
   \begin{align*}
   \dot{x} &= -y + xz^2 \\
   \dot{y} &= x + yz^2 \\
   \dot{z} &= -z(x^2 + y^2)
   \end{align*}
   \]
   has a periodic orbit $\gamma(t) = (\cos t, \sin t, 0)$. Find the linearization of this system about $\gamma(t)$, the fundamental matrix $\Phi(t)$ for the autonomous system that satisfies $\Phi(0) = I$, and the characteristic exponents and multipliers of $\gamma(t)$. What are the dimensions of the stable, unstable and center manifolds of $\gamma(t)$?

4. **Perko, Section 3.5, problem 5a**: Let $\Phi(t)$ be the fundamental matrix for $\dot{x} = A(t)x$ satisfying $\Phi(0) = I$. Use Liouville's theorem, which states that
   \[
   \det \Phi(t) = \exp \int_0^t \text{trace} A(s) ds,
   \]
to show that if \( m_j = e^{\lambda_j T}, j = 1, \ldots, n \) are the characteristic multipliers of \( \gamma(t) \) then

\[
\sum_{j=1}^{n} m_j = \text{trace} \Phi(T)
\]

and

\[
\prod_{j=1}^{n} m_j = \exp \int_{0}^{T} \text{trace} A(t) \, dt.
\]

- Hint: recall that the determinant of a matrix is equal to the product of its eigenvalues, and the trace of a matrix is equal to the sum of the eigenvalues.

5. **Perko, Section 3.9, problem 4a**: Show that the limit cycle of the van der Pol equation

\[
\begin{align*}
\dot{x} &= y + x - x^3/3 \\
\dot{y} &= -x
\end{align*}
\]

must cross the vertical lines \( x = \pm 1 \).

- Hint: you can use the fact (shown in Perko, Section 3.8) that a limit cycle exists for the van der Pol equation and that it is unique.


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