CDS140a
Nonlinear Systems: Local Theory
Lecture 3

1 The Stable Manifold Theorem

$$\dot{x} = f(x) \tag{1}$$

$$\dot{x} = Df(x_0)x \tag{2}$$

We assume that the equilibrium point x_0 is located at the origin.

1.1 Some Examples

1.1.1 Example 1

Consider The linear system

$$\begin{aligned}\dot{x}_1 &= -x_1\\ \dot{x}_2 &= 2x_2\end{aligned}$$

Clearly we have $x_1(t) = a_1 e^{-t}$ and $x_2(t) = a_2 e^{2t}$, with stable subspace $E^s = span\{(1,0)\}$ and unstable subspace $E^u = span\{(0,1)\}$. So $\lim_{t\to\infty} \phi_t(\mathbf{a}) = 0$ only if $\mathbf{a} \in R^s$. Consider a small perturbation of this linear system:

$$\begin{aligned}\dot{x}_1 &= -x_1\\ \dot{x}_2 &= 2x_2 - 5\epsilon x_1^3\end{aligned}$$

The solution is given by $x_1(t) = a_1 e^{-t}$ and $x_2(t) = a_2 e^{2t} + a_1^3 \epsilon \left(e^{-3t} - e^{2t}\right) = \left(a_2 - \epsilon a_1^3\right) e^{2t} + \epsilon a_1^3 e^{-3t}$. Clearly $\lim_{t\to\infty} \phi_t(\mathbf{a}) = 0$ only if $a_2 = \epsilon a_1^3$. Indeed we can show that the set

$$S = \{x \in \mathbb{R}^2 | x_2 = \epsilon x_1^3\}$$

is invariant with respect to the flow. It easy to see that $a_2 = \epsilon a_1^3$ leads to

$$\phi_t(S) = \begin{bmatrix} a_1 e^{-t} \\ \left(a_2 - \epsilon a_1^3\right) e^{2t} + \epsilon a_1^3 e^{-3t} \end{bmatrix} = \begin{bmatrix} a_1 e^{-t} \\ \epsilon a_1^3 e^{-3t} \end{bmatrix} \in S$$

So S is an invariant set (curve), and the flow on this curve is stable. So it seems that S is some nonlinear analog of E^s. Furthermore, notice that S is tangent to the stable subspace of the linear system, and as $\epsilon \to 0$, the curve S becomes E^s.

1.1.2 Example 2 (Perko 2.7 Example 1)_

Consider

$$\begin{aligned}\dot{x}_1 &= -x_1\\ \dot{x}_2 &= -x_2 + x_1^2\\ \dot{x}_3 &= x_3 + x_1^2\end{aligned}$$

which we can rewrite as

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ x_1^2 \\ x_1^2 \end{bmatrix}.$$

The flow is given by

$$\phi_t(S) = \begin{bmatrix} a_1 e^{-t} \\ a_2 e^{-t} + a_1^2\left(e^{-t} + e^{-2t}\right) \\ a_3 e^t + \frac{a_1^2}{3}\left(e^t - e^{-2t}\right) \end{bmatrix}$$

where $a = (a_1, a_2, a_3) = x(0)$. Clearly $\lim_{t\to\infty} \phi_t(\mathbf{a}) = 0$ only if $a_3 = -a_1^2/3$. So

$$S = \{a \in \mathbb{R}^3 | a_3 = -a_1^2/3\}$$

and similarly

$$U = \{a \in \mathbb{R}^3 | a_1 = a_2 = 0\}.$$

Again it seems that S is some nonlinear analog of $E^s$and U is some nonlinear analog of E^u . Furthermore, notice that S is tangent to the stable subspace of the linear system. We call S the stable manifold, and U the unstable manifold.
We are going to see how we can compute S and U in general.

1.2 Manifolds and stable manifold theorem

But first here is a "working" definition of a k-dimentional differential manifold. For more precise definition, there is a small section in the book, and CDS202 deals with differentiable manifolds in great details.

In this class, by **k-dimentional differential manifold** (or manifold of class C^m) we mean any "smooth" (of order C^m) k-dimensional surface in an n-dimensional space.

For example $S = \{a \in \mathbb{R}^3 | a_3 = -a_1^2/3\}$ is 2-dimensional differentiable manifold.

Theorem (The Stable Manifold Theorem): Let E be anopen subset of $\mathbb{R}^n$ containing the origin, let $\mathbf{f} \in C^1(E)$, and let ϕ_t be the flow of the non-linear system (1). Suppose that $f(0) = 0$ and that $Df(0)$ has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part. Then there exists a $k-$dimensional manifold S tangent to the stable subspace $E^s$of the linear system (2)at 0 such that for all $t \geq 0$, $\phi_t(S) \subset S$ and for all $x_0 \in S$,

$$\lim_{t\to\infty} \phi_t(x_0) = 0;$$

and there exists an $n-k$ differentiable manifold U tanget to the unstable subspace E^u of (2) at 0 such that for all $t \leq 0$, $\phi_t(U) \subset U$ and for all $x_0 \in U$,

$$\lim_{t\to-\infty} \phi_t(x_0) = 0.$$

Note: As in the examples, since $f \in C^1(E)$ and $f(0) = 0$, then system (1) can be writen as

$$\dot{x} = Ax + F(x)$$

where $A = Df(0)$, $F(x) = f(x) - Ax$, $F \in C^1(E)$, $F(0) = 0$ and $DF(0) = 0$.

Furthermore, we want to separate the stable and unstable parts of the matrix , i.e., choose a matrix C such that

$$B = C^{-1}AC = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$$

where the eigenvalues of the $k \times k$ matrix P have negative real part, and the eigenvalues of the $(n-k)\times(n-k)$ matrix Q have positive real part. The transformed system ($y = C^{-1}x$) has the form

$$\begin{aligned} \dot{y} &= By + C^{-1}F(Cy) \\ \dot{y} &= By + G(y) \end{aligned} \tag{3}$$

1.2.1 Calculating the stable manifold (Perko Method):

Perko shows that the solutions of the integral equation

$$u(t,a) = U(t)a + \int_0^t U(t-s)G(u(s,a))ds - \int_t^\infty V(t-s)G(u(s,a))ds$$

satisfy (3) and $\lim_{t\to\infty} u(t,a) = 0$. Furthermore it gives an iterative scheme for computing the solution:

$$\begin{aligned} u(t,a) &= 0 \\ u^{(k+1)}(t,a) &= U(t)a + \int_0^t U(t-s)G(u^{(k)}(s,a))ds - \int_t^\infty V(t-s)G(u^{(k)}(s,a))ds \end{aligned}$$

- **Remark** Here is some intuition on why the particular integral equation is chosen. We basically want to remove the parts that blow up as $t \to \infty$. In general, the solution of this system satisfies

$$u(t,a) = \begin{bmatrix} e^{Pt} & 0 \\ 0 & e^{Qt} \end{bmatrix} a + \int_0^t \begin{bmatrix} e^{P(t-s)} & 0 \\ 0 & e^{Q(t-s)} \end{bmatrix} G(u(s,a))ds.$$

$$\begin{aligned} u(t,a) &= \begin{bmatrix} e^{Pt} & 0 \\ 0 & e^{Qt} \end{bmatrix} a + \int_0^t \begin{bmatrix} e^{P(t-s)} & 0 \\ 0 & e^{Q(t-s)} \end{bmatrix} G(u(s,a))ds \\ & \text{Separate the convergent and non-convergent parts} \\ &= \begin{bmatrix} e^{Pt} & 0 \\ 0 & 0 \end{bmatrix} a + \begin{bmatrix} 0 & 0 \\ 0 & e^{Qt} \end{bmatrix} a + \int_0^t \begin{bmatrix} e^{P(t-s)} & 0 \\ 0 & 0 \end{bmatrix} G(u(s,a))ds + \int_0^t \begin{bmatrix} 0 & 0 \\ 0 & e^{Q(t-s)} \end{bmatrix} G(u(s,a))ds \\ &= \begin{bmatrix} e^{Pt} & 0 \\ 0 & 0 \end{bmatrix} a + \begin{bmatrix} 0 & 0 \\ 0 & e^{Qt} \end{bmatrix} a + \int_0^t \begin{bmatrix} e^{P(t-s)} & 0 \\ 0 & 0 \end{bmatrix} G(u(s,a))ds \\ & + \int_0^\infty \begin{bmatrix} 0 & 0 \\ 0 & e^{Q(t-s)} \end{bmatrix} G(u(s,a))ds - \int_t^\infty \begin{bmatrix} 0 & 0 \\ 0 & e^{Q(t-s)} \end{bmatrix} G(u(s,a))ds \\ & \text{Remove contributions that will cause it to not converge to the origin} \\ u(t,a) &= \begin{bmatrix} e^{Pt} & 0 \\ 0 & 0 \end{bmatrix} a + \int_0^t \begin{bmatrix} e^{P(t-s)} & 0 \\ 0 & 0 \end{bmatrix} G(u(s,a))ds - \int_t^\infty \begin{bmatrix} 0 & 0 \\ 0 & e^{Q(t-s)} \end{bmatrix} G(u(s,a))ds \\ &= U(t)a + \int_0^t U(t-s)G(u(s,a))ds - \int_t^\infty V(t-s)G(u(s,a))ds \end{aligned}$$

Notice that last $n-k$components of a do not enter the computation, we can take them to be zero. Next we take the specific solution $u(t,a)$

$$u(t,a) = U(t)a + \int_0^t U(t-s)G(u(s,a))ds - \int_t^\infty V(t-s)G(u(s,a))ds$$

and see what it implies with respect to the intial conditions $u(0,a)$ for the solution:

$$\begin{aligned} u_j(0,a) &= a_j, \qquad j = 1,\dots,k \\ u_j(0,a) &= -\left(\int_0^\infty V(-s)G(u(s,a))ds\right)_j, \qquad j = k+1,\dots,n \end{aligned}$$

So the last $n-k$ components of the initial conditions must satisfy

$$a_j = \psi_j(a_1,\dots,a_k) := u_j(0,a_1,\dots,a_k,0,\dots,0), \qquad j = k+1,\dots,n.$$

Therefore the stable manifold is defined by

$$S = \{(y_1,\dots,y_n) | y_j = \psi_j(y_1,\dots,y_k), \quad j = k+1,\dots,n\}.$$

- The iterative scheme for calculating an approximation to S:
  - Calculate the approximate solution $u^{(m)}(t,a)$
  - For each $j = k+1, \dots, n$, $\psi_j(a_1, \dots, a_k)$ is given by the j-th component of $u^{(m)}(0,a)$.

Note: Similarly can calculate U by taking $t = -t$.

- **Example:**

$$\begin{aligned}\dot{x}_1 &= -x_1 - x_2^2\\ \dot{x}_2 &= x_2 + x_1^2\end{aligned}$$

$$A = B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad F(x) = G(x) = \begin{bmatrix} -x_2^2 \\ x_1^2 \end{bmatrix}$$

$$U = \begin{bmatrix} e^{-t} & 0 \\ 0 & 0 \end{bmatrix}, V = \begin{bmatrix} 0 & 0 \\ 0 & e^{t} \end{bmatrix}, a = \begin{bmatrix} a_1 \\ 0 \end{bmatrix}$$

Then

$$\begin{aligned}
u^{(0)}(t,a) &= \begin{bmatrix} a_1 \\ 0 \end{bmatrix}\\
u^{(1)}(t,a) &= \begin{bmatrix} e^{-t}a_1 \\ 0 \end{bmatrix}\\
u^{(2)}(t,a) &= \begin{bmatrix} e^{-t}a_1 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-(t-s)} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ e^{-2s}a_1^2 \end{bmatrix} ds - \int_t^\infty \begin{bmatrix} 0 & 0 \\ 0 & e^{(t-s)} \end{bmatrix} \begin{bmatrix} 0 \\ e^{-2s}a_1^2 \end{bmatrix} ds = \begin{bmatrix} e^{-t}a_1 \\ -\frac{e^{-2t}}{3}a_1^2 \end{bmatrix}\\
u^{(3)}(t,a) &= \begin{bmatrix} e^{-t}a_1 + \frac{1}{27}(e^{-4t} - e^{-t})a_1^4 \\ -\frac{e^{-2t}}{3}a_1^2 \end{bmatrix}
\end{aligned}$$

Next can show that $u^{(34)}(t,a) - u^{(3)}(t,a) = O(a_1^5)$ and therefore we can approximate by $\psi_2(a_1) = -\frac{1}{3}a_1^2 + O(a_1^5)$ and the stable manifold can be approximated by

$$S \,:\, x_2 = -\frac{1}{3}x_1^2 + O(x_1^5)$$

as $x_1 \to 0$. Similarly get

$$U \,:\, x_1 = -\frac{1}{3}x_2^2 + O(x_2^5)$$

1.2.2 Note on invariant manifolds:

Notice that if a manifold is specified by a constraint equation

$$y = h(x), \qquad x \in \mathbb{R}^k, y \in \mathbb{R}^{n-k}$$

and the dynamics given by

$$\begin{aligned}\dot{x} &= f(x,y)\\ \dot{y} &= g(x,y)\end{aligned}$$

then condition

$$\begin{aligned} Dh(x)\dot{x} &= \dot{y}\\ &\Downarrow\\ Dh(x)f(x,h(x)) &= g(x,h(x))\end{aligned}$$

suffices to show invariance. We'll call this tangency condition. Exercise: Show that this is the case. *If you're going to use this in the homework this week, you should prove it first.*

- ***Example:***

$$\begin{aligned}\dot{x}_1 &= -x_1\\ \dot{x}_2 &= 2x_2 - 5\epsilon x_1^3\end{aligned}$$

Show that the set

$$S = \{x \in \mathbb{R}^2 | x_2 = \epsilon x_1^3\}$$

is invariant. We have

$$3\epsilon x_1^2(-x_1) = 2\epsilon x_1^3 - 5\epsilon x_1^3.$$

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1.2.3 Calculating the stable manifold (Alternative Method - Taylor expansion):

Let

$$y = h(x) = ax^2 + bx^3 + cx^4 + \dots$$

Since invariant manifold we have:

$$Dh(x)\dot{x} - \dot{y} = 0$$

we can match coefficients. For example

$$\begin{aligned}\dot{x}_1 &= -x_1\\ \dot{x}_2 &= 2x_2 - 5\epsilon x_1^3\end{aligned}$$

$$x_2 = h(x_1) = ax_1^2 + bx_1^3 + O(x_1^4)$$

we get $f(x_1, h(x_1)) = -x_1$, $g(x_1, h(x_1) \approx 2(ax_1^2 + bx_1^3) - 5\epsilon x_1^3$

$$\begin{aligned}Dh(x)f(x, h(x)) &= g(x, h(x))\\ &\Downarrow\\ (2ax_1 + 3bx_1^2 + \cdots)(-x_1) &= 2ax_1^2 + 2bx_1^3 - 5\epsilon x_1^3 +\end{aligned}$$

Matching terms we get $-2a = 2a \Rightarrow a = 0$, $-3b = 2b - 5\epsilon \Rightarrow b = \epsilon$.

1.3 Global Manifolds

- In the proof S and U are defined in a small neighborhood of the origin, and are refered to as the *local* stable and unstable manifolds of the origin.

Definition: Let ϕ_t be the flow of (1). The *global stable* and *unstable manifolds* of (1) at 0 are defined by

$$W^s(0) = \cup_{t \leq 0} \phi_t(S)$$

and

$$W^u(0) = \cup_{t \geq 0} \phi_t(S)$$

respectively.

The global stable and unstable manifold $W^s(0)$ and $W^u(0)$ are unique and invariant with respect to the flow. Furthermore, for all $x \in W^s(0)$, $\lim_{t\to\infty} \phi_t(x) = 0$ and for all $x \in W^u(0)$, $\lim_{t\to-\infty} \phi_t(x) = 0$.

Corollary: Under the hypothesis of the Stable Manifold theorem, if $Re(\lambda_j) < -\alpha < 0 < \beta < Re(\lambda_m)$ for $j = 1, \dots, k$ and $m = k+1, \dots, n$ then given $\epsilon > 0$,there exists a $\delta > 0$such that if $x_0 \in N_\delta(0) \cap S$ then

$$|\phi_t(x_0)| \leq \epsilon e^{-\alpha t}$$

for all $t \geq 0$and if $x_0 \in N_\delta(0) \cap SU$ then

$$|\phi_t(x_0)| \leq \epsilon e^{\beta t}$$

for all $t \leq 0$.