CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 110b
Midterm Solutions

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1. Estimators

2. Stochastic Systems

   (a) Since we care about steady state properties, we can ignore the transients due to the initial condition. The output y is then given by

   $$y(t) = \int_0^t e^{-b(t-\xi)} v(\xi)\, d\xi + dv(t).$$

   To simplify the derivation, we write

   $$y(t) = x_2(t) + dv(t) \qquad x_2 = \int_0^t e^{-b(t-\xi)} v(\xi)\, d\xi.$$

   The correlation function is given by

   $$\begin{aligned} \rho(\tau) &= E\{y(t)y(t+\tau)\} \\ &= E\{\big(x_2(t) + dv(t)\big) \cdot \big(x_2(t+\tau) + dv(t+\tau)\big)\} \\ &= E\{x_2(t)x_2(t+\tau)\} + dE\{x_2(t)v(t+\tau)\} + dE\{v(t)x_2(t+\tau)\} + d^2 E\{v(t)v(t+\tau)\}. \end{aligned}$$

   The first term is just the correlation function of x_2, which is a first order filter applied to white noise. This is a simplified version of the problem from homework #2, problem 2 and is given by

   $$\rho_{x_2}(\tau) = \frac{\sigma^2}{2b} e^{-b\tau} \qquad \tau > 0.$$

   Similarly, the last term (the "direct" term) is just the correlation function for the noise term

   $$\rho_v(\tau) = \sigma^2 \delta(\tau).$$

   The cross terms are computed by bringing everything inside the integrals:

   $$\begin{aligned} E\{x_2(t)v(t+\tau)\} &= \int_0^t e^{-b(t-\xi)} E\{v(\xi)v(t+\tau)\}\, d\xi &&= \sigma^2 e^{b\tau} \\ E\{v(t)x_2(t+\tau)\} &= \int_0^{t+\tau} e^{-b(t+\tau-\xi)} E\{v(\xi)v(t)\,\} d\xi &&= \sigma^2 e^{-b\tau}, \end{aligned}$$

   where we have taken $\tau > 0$. Thus the correlation function is given by

   $$\rho(\tau) = \sigma^2 \left(\frac{1}{2b} e^{-b|\tau|} + de^{b|\tau|} + de^{-b|\tau|} + d^2 \delta(\tau) \right)$$

(the absolute values follow form the fact that for $\tau < 0$ we get the same answer by shifting the time axis forward, $t' = t + \tau$). This works because we care only about steady state properties.

(b) The mean of the output is 0 since the noise term has zero mean. The covariance of the output is determined by setting $\tau = 0$ in $\rho(\tau)$ and is unbounded.

3. Kalman Filtering

First of all, the system is scalar; the noise covariances are unitary.

(a) The Riccati equation for the state estimation error is:

$$\dot{p} = 2\lambda p - \frac{p^2}{\sigma_w} + \sigma_v$$

For $\sigma_w = \sigma_v = 1$ it simplifies to

$$\dot{p} = 2\lambda p - p^2 + 1$$

Finding its steady state solution:

$$0 = 2\lambda p - p^2 + 1$$

leads to $p^*_{1,2} = \lambda(1 \pm \sqrt{\lambda^2 + 1})$. Since this is a covariance, it has to be positive. Therefore we only keep the positive solution $p^* = \lambda(1 + \sqrt{\lambda^2 + 1})$.

(b) Here we linearize the Riccati equation about its equilibrium point.

$$\begin{aligned} \delta\dot{p} &= 2(\lambda - \delta p^*)\delta p \\ &= 2(\lambda - \lambda - \sqrt{\lambda^2 + 1})p \\ &= -2(\sqrt{\lambda^2 + 1})p \end{aligned}$$

The eigenvalue of this equation is therefore $-2(\sqrt{\lambda^2 + 1})$, which is negative regardless of the value of λ. The equation will therefore always converge.

(c) In this case we need to define an auxiliary system that accounts for the colored noise. A choice is:

$$\dot{w} = \alpha w + q$$

where now q is WGN with a certain covariance $Q = 2\alpha$. If we compute the correlation for $w(t)$, we will obtain the given $\rho_w(\tau) = e^{-\alpha|\tau|}$.
Now we can consider new state and measurement: $X = col(x, w)$, $Y = CX$, where $C = [\ 1 \quad \sigma_w\ ]$. Note that Y is still affected by colored noise.
From now on we can follow Friedland, Section 11.6. Define:

$$\begin{aligned} z &= \dot{Y} - \alpha Y = \dot{C}X + C\dot{X} - \alpha CX \\ &= \lambda x + u + \sigma_v v + \sigma_w(\alpha w + q) - \alpha x - \sigma_w \alpha w, \\ &= hx + u + n. \qquad \text{where}\, h := \lambda - \alpha,\ n := \sigma_v v + \sigma_w q. \end{aligned}$$

If we took this as our measurement, we would now have the problem of measurement noise correlated with process noise:

$$E[v\,n^T] = \sigma_v, \qquad E[n\,n^T] = \sigma_v^2 + 2\alpha\sigma_w^2 := N.$$

We can solve the problem by noting that $z - (Hx + u + n) = 0$, and we can thus add this quantity to the old state dynamics through a Lagrange multiplier that will help us to "uncorrelate" the process and measurement noises.

$$\begin{aligned}\dot{x} &= \lambda x + u + \sigma_v v + \nu(z - hx - u - n)\\ &= (\lambda - \nu h)x + (1 - \lambda)u + \sigma_v v - \nu n + \nu z.\end{aligned}$$

By suitably choosing ν we can get what we are looking for:

$$E[(\sigma_v v - \nu n)\,n^T] = \sigma_v^2 - \nu\,N = 0 \Leftrightarrow \nu := \sigma_v^2\,N^{-1}$$

With the above choices, the equations for the filter can be defined as:

$$\begin{aligned}\dot{\hat{x}} &= (\lambda - \nu h)\hat{x} + K(z - h\hat{x}) + \nu z + (1 - \nu)u\\ K &= p\,h\,N^{-1}\end{aligned}$$

and therefore we can simplify to:

$$\begin{aligned}\dot{\hat{x}} &= \lambda\hat{x} + \bar{K}(z - h\hat{x}) + (1 - \nu)u\\ \bar{K} &:= K + \nu.\end{aligned}$$

The corresponding equation for the state estimation error covariance is:

$$\dot{p} = (\lambda - \nu h)^2 p + \sigma_v^2 - \nu^2 N - p^2\,h^2 N^{-1}$$

Note: this process requires differentiation of the measurement, which one should always avoid. There are ways to move the problem to differentiation of the Kalman gain.

4. Discrete Time Kalman Filters

Note that the processes are identical in x and y, so it is sufficient to compute the solution for one variable.

(a) The solution for the process at time k is given by

$$x[k] = x[0] + \sum_{i=0}^{k-1} u[i].$$

The expected value is

$$E\{x[k]\} = E\{x[0]\} + \sum_{i=0}^{k-1} E\{u[i]\} = E\{x[0]\} =: \mu_x.$$

The (co-)variance is given by $E\{(x[k]-\mu_x)^2\}$, where we have subtracted off the mean since it is nonzero. Since the input is a white noise random process, it follows that

$$E\{(x[k]-\mu_x)^2\} = \sum_{i=0}^{k-1} E\{u^2[i]\} = kR_u$$

(all of the cross terms between the u's disappear).

Note: in the problem, it incorrectly asked to compute the steady state properties (it was supposed to have asked for the expected value and covariance as a function of k). You were given full credit if you properly showed (along the lines of the calculation above) that the steady state covariance was unbounded.

(b) The Kalman filter for the process is given by

$$\hat{x}[k+1] = \hat{x}[k] + L[k](x[k]-\hat{x}[k]) \qquad P[k+1] = (1-L[k])^2 P[k] + R_u + L^2[k] R_w$$
$$L[k] = P[k](R_w + P[k])^{-1} \qquad P[0] = E\{x^2[0]\}.$$

where $R_w = 1$ is the covariance of the measurement noise. This equation follows directly from Åström and Murray using $A = 1$, $F = 1$, $C = 1$ and simplifying due to the fact that all quantities are scalars. You could also write down the equivalent two stage equations. The steady state properties of the *error*, e, are given by $E\{e\} = 0$ and $E\{e^2\} = P$, where P satisfies:

$$P = APA^T + R_u - APC^T\left(R_w + CPC^T\right)^{-1} CPA^T.$$

(AM06, page 231; can also be worked out from predictor-corrector form). Substitution of the values of $A = 1$ and $C = 1$ give

$$P^2 - R_u P - R_u R_w = 0$$

which can be solved using the quadratic formula for the positive root:

$$P_+ = \frac{1}{2}R_u + \frac{1}{2}\sqrt{R_u^2 + 4R_u R_w}.$$

Note: the problem incorrectly asked for the expected value and covariance of the *estimate*, rather than the error. You were given partial credit if you stated that the covariance would be unbounded (since the covariance of the original system is unbounded).

(c) If the mean of u is nonzero, then you get a biased random walk. In this case, the mean and variance of the process are given by

$$E\{x[k]\} = E\{x[0]\} + k\mu$$
$$E\{(x[k]-\mu_x)^2\} = k + k^2$$

(follow from direct computation). The Kalman filter can be updated to take into account the (known) mean of u and hence this term can be subtracted out. The filter equations become

$$\hat{x}[k+1] = \hat{x}[k] + \mu + L[k](x[k]-\hat{x}[k]).$$

The statistics for the error are unchanged because we subtract off the mean.