ODE / Linear Algebra Review (Lecture 2.3)

We have seen that we design and analyze feedback systems based on a model of the system

$$\text{Input} \longrightarrow \boxed{\text{System}} \longrightarrow \text{output}$$

Very often we use differential equations to describe the input/output behavior. Many systems we'll look at this term will take the form of linear state space systems:

$$(*) \qquad \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \qquad \left.\begin{array}{ll} x \in \mathbb{R}^n & \text{"state"} \\ u \in \mathbb{R}^m & \text{"input"} \\ y \in \mathbb{R}^p & \text{"output"} \end{array}\right\} \text{ vectors!}$$

The dynamics and input/output behavior of the system are defined by matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$. The analysis of such a system is performed through linear algebra and the study of ODEs of form $(*)$.

→ Linear Algebra Notes go here

Linear Differential Equations

1. If you have a system of first order differential equations it can be written as $(*)$

e.g.
$$\dot{z}_1 = 4z_1 - 3z_2 + 7u$$
$$\dot{z}_2 = z_1$$
$$\downarrow$$
$$\underbrace{\frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}}_{\dot{x}} = \underbrace{\begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}}_{x} + \underbrace{\begin{pmatrix} 7 \\ 0 \end{pmatrix}}_{B} \underbrace{u}_{u}$$

Linear Algebra

Consider $A \in \mathbb{R}^{n \times n}$

A vector $v \in \mathbb{R}^n$ is an eigenvector of A with corresponding eigenvalue $\lambda \in \mathbb{C}$ if $Av = \lambda v$

Note: real matrices can have complex eigenvalues, but they always occur in conjugate pairs. $\lambda, \bar{\lambda}$.

Finding eigenvalues:

$$\det(A - \lambda I) = 0$$

Solve for λ.

Finding eigenvectors:

$$(A - \lambda_i I) v_i = 0 \qquad \text{for } i = 1 \ldots n$$

Upper or Lower Triangular matrices: eigenvalues are along the diagonal:

e.g. $\begin{pmatrix} 4 & 3 \\ 0 & -6 \end{pmatrix}$ $\quad \lambda_{1,2} = 4, -6$ (candy bar example)

Matlab: eig

Aside: Positive Definite Matrices

P is positive definite if $x^T P x > 0$ for all $x \neq 0$

this is written as $P > 0$ and does not mean that all elements are positive!

Negative definite: $x^T P x < 0$

Negative semi-definite: $x^T P x \leq 0$

Positive semi-definite $x^T P x \geq 0$

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2. If you have a general scalar differential equation it can also be written as (*)

only include if time

Let $z^{(i)} = \frac{d^i z}{dt^i}$ be the ith derivative of z w.r.t. t.

$$z^{(n)} + p_{n-1} z^{(n-1)} + \dots p_1 z^{(1)} + p_0 z = u$$

let

$$z_1 = z \qquad \dot{z}_1 = z_2$$
$$z_2 = \dot{z} \qquad \text{and notice} \qquad \dot{z}_2 = z_3$$
$$z_3 = \ddot{z} \qquad \vdots$$
$$\vdots \qquad \dot{z}_{n-1} = z_n$$
$$z_n = z^{(n-1)} \qquad \dot{z}_n = -p_{n-1} z^{(n-1)} \dots -p_0 z + u$$
$$= -p_{n-1} z_n \dots -p_0 z_1 + u$$

$$\frac{d}{dt}\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} \begin{matrix} 0 \\ | \\ 0 \end{matrix} & \begin{bmatrix} & I & \end{bmatrix} \\ -p_0 & -p_1 \; -- \; -p_{n-1} \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} u$$

$$\dot{x} = A \quad x + B u$$

e.g. mass-spring system

$$m\ddot{z} + b\dot{z} + Kz = u$$

let $z_1 = z$ then $\dot{z}_1 = z_2$

$z_2 = \dot{z}$

$$\dot{z}_2 = \ddot{z} = \frac{1}{m}(-b\dot{z} - Kz + u)$$
$$= \frac{1}{m}(-bz_2 - Kz_1 + u)$$

$$\frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{K}{m} & -\frac{b}{m} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} u$$

$$\dot{x} = A \quad x + B u$$

Solving (*)

Simplest example : $n=1$, $u(t)=0$ (scalar case)

$$\dot{x} = ax$$

Solution $\rightarrow$ $x(t) = e^{at}x(0)$

Check : $\frac{d}{dt}\left(e^{at}x(0)\right) = a e^{at} x(0) = a e^{at} x(0)$ ✓

More general (matrix case)

$$\dot{x} = Ax$$

Solution $\rightarrow$ $x(t) = e^{At}x(0)$

Matrix Exponential

Recall series definition $e^x = 1 + x + \frac{1}{2}x^2 + \dots \frac{1}{k!}x^k + \dots$

Similarly $e^{At} = I + At + \frac{A^2t^2}{2!} + \dots = \sum_{i=0}^{\infty} \frac{A^it^i}{i!}$

$(A^2 = AA,\ A^3 = AAA, \dots)$

$$\frac{d}{dt}\left(e^{At}\right) = \frac{d}{dt}\left(I + At + \frac{A^2t^2}{2!} + \dots\right)$$

$$= 0 + A + A^2t + \dots$$

$$= A\left(I + At + \dots\right)$$

$$= Ae^{At}$$

Remarks

1. $e^A e^{-A} = I$
2. $e^A e^B \neq e^{A+B}$ unless $AB = BA$
3. $e^{TAT^{-1}t} = \underset{TT^{-1}}{\overset{\shortparallel}{I}} + TAT^{-1}t + \frac{(TA\cancel{T^{-1}})(\cancel{T}AT^{-1})t^2}{2!} + \dots$

$$= T\left(I + At + \frac{A^2t^2}{2!} + \dots\right)T^{-1}$$

$$= Te^{At}T^{-1}$$

4. If A is diagonal (candy bar question)

$$e^{\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}t} = \begin{bmatrix}e^{\lambda_1 t} & & 0\\ & \ddots & \\ 0 & & e^{\lambda_n t}\end{bmatrix}$$

5. expm in Matlab

So solution to $\dot{x} = Ax$ is $x = e^{At} x(0)$

In general the solution to $\dot{x} = Ax + Bu$ is

$$\boxed{x(t) = \underbrace{e^{At} x(0)}_{\text{homogeneous solution}} + \underbrace{e^{At} \int_0^t e^{-A\tau} B u(\tau)\, d\tau}_{\text{particular solution}}}$$

→ HW #1 prob 1 this is one way to solve

Check $\dot{x} = A e^{At} x(0) + \left(A e^{At}\right)\left(\int_0^t e^{-A\tau} B u(\tau) d\tau\right)$

$$+ \underbrace{\left(e^{At}\right)\left(e^{-At}\right.}_{= I}\left. B u(t)\right)$$

$$= A\left(e^{At} x(0) + e^{At} \int_0^t e^{-A\tau} B u(\tau) d\tau\right)$$

$$+ Bu$$

$$= Ax + Bu \quad \checkmark$$

Note that as $t \to \infty$ e^{At} will determine if $x(t)$ becomes small or blows up (stability of the system).

Can always find a transformation $A = TJT^{-1}$ where T is a matrix of eigenvectors of A $T = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$ and J is in Jordan Form. More on this later in class.

If the e-vectors of A are linearly independent can always find a transformation $A = T \Lambda T^{-1}$ where $\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ λ_i are e-values of A

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Behavior of e^{At} as $t \to \infty$:

$$e^{At} = e^{T \Lambda T^{-1} t} = T e^{\Lambda t} T = T \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix} T^{-1}$$

→ if <u>any</u> of the eigenvalues of A have positive real parts then $x(t)$ will grow as $t \to \infty$.

(candy bar question: what properties does the matrix A need to have to be a stable system?