CDS 140a Winter 2014 Homework 4

From MurrayWiki

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ACM 101b/AM 125b/CDS 140a, Winter 2014 — Due: 4 Feb 2014 (Wed) @ noon

Turn in to box outside Steele House

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. **Perko, Section 2.7, problem 2** Find the first three successive approximations $u^{(1)}(t,a)$, $u^{(2)}(t,a)$, and $u^{(3)}(t,a)$ for

$$\dot{x}_1 = -x_1,$$
$$\dot{x}_2 = x_2 + x_1^2$$

and use $u^{(3)}(t,a)$ to approximate S near the origin. Also approximate the unstable manifold U near the origin for this system. Note that $u^{(2)}(t,a) = u^{(3)}(t,a)$ and therefore $u^{(j+1)}(t,a) = u^{(j)}(t,a)$ for $j \geq 2$. Thus $u(t,a) = u^{(2)}(t,a)$ which gives the exact function defining S.

2. **Perko, Section 2.7, problem 3** Show that S and U for the previous problem are given by

$$S:\ x_2 = -\frac{x_1^2}{3}$$
$$U:\ x_1 = 0$$

Sketch S, U, E^s and E^u.

3. Consider a dynamical system with $x = (u, v) \in \mathbb{R}^n$. For the case $n = 2$, prove that if

$$\dot{u} = f(u,v), \qquad u \in \mathbb{R}^k$$
$$\dot{v} = g(u,v), \qquad v \in \mathbb{R}^{n-k}$$

then the manifold $S = \{(u,v) \in \mathbb{R}^k \times \mathbb{R}^{n-k} | v = h(u)\}$ is an invariant manifold of the system if

$$g(u, h(u)) = Dh(u) f(u, h(u))$$

Use this result to compute the stable manifold for the system of problem 1 and 2 above using the Taylor series for $h(x)$ to define $S = \{(x_1, x_2) | x_2 = h(x)\}$ and matching coefficients to solve for $h(x)$.

- Note: The result holds for $\mathbb{R}^n$, but you only need to consider the case $n = 2$ and $k = 1$ (although you are free to prove the more general case if you prefer).
- Hint: One way to show S is an invariant manifold in $\mathbb{R}^2$ is to show that the normal vector (orthogonal to the tangent to S at $(x, h(x))$) is orthogonal to the vector field (f, g) at that point. (It is sufficient to prove the result for $\mathbb{R}^2$.)

4. **Perko, Section 2.6, problem 3** Show that the continuous map $H : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = H(x) = \begin{bmatrix} x_1 \\ x_2 + x_1^2 \\ x_3 + x_1^2/3 \end{bmatrix}$$

has a continuous inverse $H^{-1} : \mathbb{R}^3 \to \mathbb{R}^3$ and that the nonlinear system

$$\frac{dx}{dt} = \begin{bmatrix} -x_1 \\ -x_2 + x_1^2 \\ x_3 + x_1^2 \end{bmatrix}$$

is transformed into a linear system using this transformation, i.e., if $y = H(x)$, show that $\dot{y} = Ax$ for an appropriate A. Use this transformation to compute and sketch the stable and unstable manifolds for the nonlinear system.

- Note: since the nonlinear system can be transformed into a linear one, it follows that $S = H^{-1}(E^s)$ and $U = H^{-1}(E^u)$.

5. **Perko, Section 2.8, Problem 1** Solve the system

$$\begin{aligned} \dot{y}_1 &= -y_1 \\ \dot{y}_2 &= -y_2 + z^2 \\ \dot{z} &= z \end{aligned}$$

and show that the successive approximations $\Phi_k \to \Phi$ and $\Psi_k \to \Psi$ as $k \to \infty$ for all $x = (y_1, y_2, z) \in \mathbb{R}^3$. Define $H_0 = (\Phi, \Psi)^T$ and use this homeomorphism to find

$$H(x) = \int_0^1 e^{-As} H_0\big(\phi_s(x)\big)\, ds,$$

where A is the linearization of the nonlinear dynamics at the origin and $\phi_t(x)$ is the flow of the full system. Use the homemorphism H to find the stable and unstable manifolds

$$W^s(0) = H^{-1}(E^s) \quad \text{and} \quad W^u(0) = H^{-1}(E^u)$$

for this system.

- Note: this problem involves some simple but somewhat tedious computations. If you know how to use Mathematica or a similar program, you may wish to carry out the computations for the successive approximations using that software. However, make sure to show the results at each step of the calculation.
- Hint: You should find

$$H(y_1, y_2, z) = \begin{bmatrix} y_1 \\ y_2 - z^2/3 \\ z \end{bmatrix}, \qquad \begin{aligned} W^s(0) &= \{x \in \mathbb{R} | z = 0\}, \\ W^u(0) &= \{x \in \mathbb{R} | y_1 = 0, y_2 = z^2/3\}. \end{aligned}$$

Notes:

- The problems are transcribed above in case you don't have access to Perko. However, in the case of discrepancy, you should use Perko as the definitive source of the problem statement.